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RECREATIONAL MATHEMATICS • TILINGS
PERMUTATIONS • ANNIHILATION GAMES

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COVER: M. C. Escher's "Shells and Starfish" contains the fundamental pentagonal tile from which this cover was built. A block of four such pentagons on page 29 illustrates but one of the symmetries discussed there in an article on pentagonal tilings of the plane. Escher's drawings are reproduced by permission of the Escher Foundation, Haags Gemeentemuseum, The Hague.

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ABOUT OUR AUTHORS

John Horton Conway ("A Gamut of Game Theories") of Cambridge University has done much to bring games to the attention of both the public and mathematicians. Martin Gardner's column in *Scientific American* has frequently reported on Conway inventions. In addition, Conway is nearly as well known for a book he didn't write (*Surreal Numbers* by Donald Knuth) as for one he did write (*On Numbers and Games*). The interplay between "surreal" numbers and game theory represents one of Conway's most imaginative contributions to current mathematics.

Aviezri S. Fraenkel, Uzi Tassa and **Yaacov Yesha** ("Three Annihilation Games") are from The Weizmann Institute of Science in Rehovot, Israel. They are interested in combinatorial games, combinatorics, complexity studies, number theory and information retrieval. Tassa did his M.Sc. thesis on the game "Battle of Numbers", and is presently working on partisan games for his Ph.D. thesis. Yesha is completing a Ph.D. thesis on "annihilation games". These studies, and other work on games at The Weizmann Institute of Science, are aimed at finding polynomial strategies for increasingly sophisticated combinatorial games, or, failing this, proving them "complex" (NP-hard).

Charles W. Trigg ("What is Recreational Mathematics?") started his professional career as an engineer with a special interest in the production of instant coffee. He then turned to teaching in high school and college and is currently Professor and Dean Emeritus at Los Angeles City College. He has always been active in problem solving and has edited the problem sections of several journals. He introduced the popular Quickies section of this Magazine.

Samuel Yates ("The Mystique of Repunits") studied engineering at the Cooper Union Institute of Technology, majored in mathematics at the George Washington University, and worked for advanced degrees in computer and information sciences at the University of Pennsylvania. His employment has been entirely in related practical applications fields, but his current principal avocation is research in periodicity and repunits. In addition to writing on these subjects, he has published a table of period lengths of the first 105,000 primes, and he maintains and updates a list of all known primitive prime divisors of the first 10,000 repunits.

Doris Schattschneider ("Tiling the Plane with Congruent Pentagons") received her Ph.D. from Yale University in 1966. After teaching at Northwestern University and the University of Illinois at Chicago Circle, she came to Moravian College in 1968. Here her interest in art led her to create a January Term course "Tessellations: The Mathematical Art". Interest in the problem she writes about here was a natural corollary. Recently she has collaborated with a graphic artist to produce a book and collection of unique geometric models, *M. C. Escher Kaleidocycles*, Ballantine Books. The article is greatly expanded from a talk given at Miami University in 1976, in order to include the most current information.

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A Gamut of Game Theories

The evaluation of compounds of games, playing lots of games at once, gives a natural construction of surreal numbers.

JOHN HORTON CONWAY
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I'm here to talk about a number of different kinds of theories of games. The basic idea here is that instead of just taking *one* game and analyzing it and seeing who wins — which is quite an interesting thing to do in itself — you try to define something that will work for a whole lot of games at once. This enables you (sometimes) to take a new game you may never have heard of before, immediately sit down, doodle something on a piece of paper, and say, "OK, Left wins, but only if he makes *this* move; if he makes any other move, Right has a win in at most 42 moves". Something like that.

Most of these theories concern compounds of games. The basic idea is that you have a whole lot of games, $G, H, \ldots$ and somehow you get some wonderful compound which might be called $G + H + \cdots$ (that's when you're sort of adding them up) or — I'll just put a $*$ here — $G * H * \cdots$.

Now imagine that we have a big table. (It has to be quite a big table, because several games are going on at once.) There are two players; and there is game G, and there is game H, and there may be more. Somehow each of these two players must move in the compound game. Now that may mean making a move in just one of G and H; it may mean making a move in both. It may mean choosing whether you move in one or both. It can be all sorts of things. And then you have different rules to see exactly what it means to win the compound game. You try to see, "What need we to know about $G, H, \ldots$ to win $G * H * \cdots$?"

That's the basic idea of this theory. It may not be sufficient to know just who wins the little games — you may need to know a lot more. What you need to know doesn't just depend on the games, $G, H, \ldots$, but it depends on the way you combine them to form this compound game. And there are a whole slew of different answers.

One of the invited lectures from the conference on Recreational Mathematics held at Miami University, Oxford, Ohio, September 24–25, 1976, this paper is an edited transcript of Conway's talk prepared by Paul J. Campbell of Beloit College. Other papers from the conference appear in this issue, together with a few papers on similar themes that were not part of that conference.

The (disjunctive) sum of games

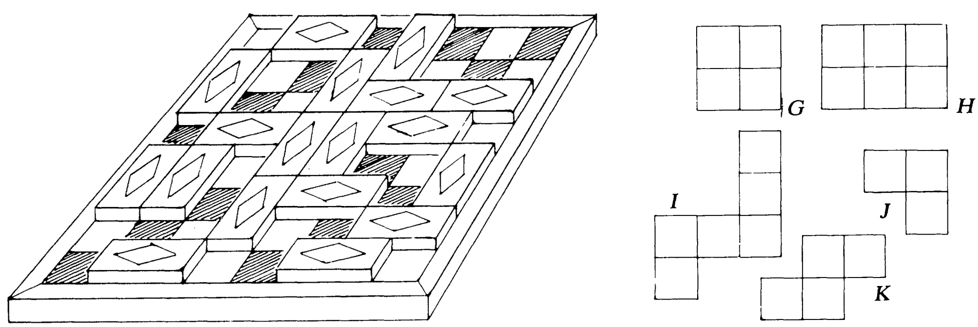
Let's talk about $+$. This is the most important case. What do I mean by $+$? The (disjunctive) **sum** of games — let's call it $G + H + \cdots$ — is what you get if you move in it by making a legal move in *just one* of the components $G, H, \ldots$.

In what I shall call **normal play,** the basic rule is this: If you can't move, you lose. This is a very good rule all the way through, and you're bound to forget it, so I shall say it seventeen times in the course of this lecture. It's sufficient to define the end of the game. You just carry on playing the game. Our games are going to have some kind of terminating condition usually, so that they have to end. (Although maybe not: the whole point about having a lot of different theories is that you can take any restriction applied to one theory and say, "Oh well, we're not going to apply that any more", and see what the more interesting theory is.) What I should, of course, say to make this a bit more precise is, if you can't make a legal move, you lose; whereupon, if you like, you can just sort of turn the whole board over and walk away in disgust.

Let's have a quick look at how we can apply this sort of thing. We could have a domino game. Here is "Domineering", proposed by Goran Andersson. You have a checkerboard, and there are two players, called Left and Right. (Actually, Right is normally thought of as female in all these games; her name is sometimes Rita and he's often called Lefty.) Left puts a vertical domino in, and Right puts a horizontal domino in. Who wins? Well, I've already told you that. Second time, now: If you can't move, you lose. So the first person who's unable to find enough space to put his domino loses.

You just carry on putting your dominoes in. The dominoes aren't allowed to overlap each other or to hang off the edge of the board — well, they may be, but that's a different game and while the same arguments would apply, you'd get different answers.

Now why is this kind of theory going to be of any use to us? Well, it's going to be of use because already I've started cutting this board up. You'll see that the board (FIGURE 1) is now divided into a number of regions, say G, H, I, J, K. When you make your move, you're just forced to choose one of those regions and put your domino in it. You can't possibly put your domino in two regions at once, by the rules of the game. So, in fact, you are playing the sum $G + H + \cdots + K$, where by G I really mean the game you'd get if you supposed that the only space left was that 2×2 square region in the northwest corner. You're just *forced* to choose one of the component games $G, H, \ldots, K$ and make a move in it that is legal for you.



In the game of "Domineering" Rita must place dominoes horizontally and Lefty must play vertically on a checkerboard whose spaces are each one half of a domino. In this half-played game, the regions remaining (pictured to the right of the board) are separated; so the game has split into a sum of several simultaneous games.

FIGURE 1.

Sums of impartial games and the Grundy-Sprague theory

The Grundy-Sprague theory is useful for what we call impartial games. An **impartial** game is one in which any move legal for you is legal for me. The domino game is not impartial; it's **a partisan** game, and that means that the moves are different for the two players. Left must place a vertical domino, Right must place a horizontal domino, so a move that's legal for Left isn't legal for Right. Similarly in chess, for instance: if you're the player called Black, you only move Black pieces, and it's not legal for you to move a White one.

For impartial games, the Grundy-Sprague theory says: *They're all like Nim.* I'll just give a very quick description of what Nim is. In Nim, you play with a number of heaps — of beans or cowrie shells or something — and the move is to reduce the size of any heap by taking some things away from it. You've got a number of heaps, and you take some objects off just one of the heaps. You can choose which one, and you can choose how many things you take off it. You must take *some* things off just one of the heaps. Then I choose a heap and take some things off it, and so on. And whoever can't move loses. When can't you move? When there are no objects left.

There's a sort of little theory concerned with adding up Nim heaps by a strange kind of addition. If I use $*n$ to mean a heap of size n, then if you have a Nim heap of size 3 and a Nim heap of size 5, they together count as a Nim heap of size 6. So you can pretend that everything is just a single heap — the whole game is just one heap — because you can add games in this funny way.

So what happens now? Suppose I'm playing a one-heap game, what do I do in order to win this game? If it's just one heap, it's very easy: I just choose that heap to move from (I'm free to do that) and I take everything. Then I call upon you to make your move, and of course you can't move, so you lose.

The basic idea is to move so that you leave heaps which sum to zero. It turns out, for instance, that $*n + *n$ is always zero: $*n + *n = *0$. So if you move so as to leave two equal heaps, then you've really moved to a heap of size zero. It may not look like it, but it is! If you've got two equal heaps, I call on you to move, and this time you really can make a move (so you think!). Let's suppose we have a heap of size 17 and another heap of size 17. You reduce one heap to size 10, I'll reduce the other to size 10. You make a further reduction of one of these things to size 5, I make another one. In the end, you will reduce one of them to zero, I'll reduce the other one to zero; and then you can't move, so you lose.

That theory is a very popular theory, very well known, and very well understood. There is an extension of it due to C.A.B. Smith, as far as I know. This allows non-terminating games. I'm not going to say any more about it, except to give you a nice example.

Suppose one has a number of heaps. We can play the game called "Fair Shares or unequal Partners". The move in this game is to divide any one of those heaps into several smaller heaps, all of *equal sizes* (that's the "fair shares" bit), or to unite any number of heaps, provided they're all of *different sizes* (that's the "unequal partners" bit). When does the game end? Well, suppose that the whole thing has been split into a whole lot of heaps all of size 1. Then you can't move, because you can't split such a heap into equal-size heaps; nor can you unite anything, because you're only allowed to unite unequal-size heaps. And so the game ends when everything is split into heaps of size 1, because then you can't move. That's quite an interesting example of the theory, for you can play the sum of several copies of it by having, say, a number of heaps of white beans and another number of heaps of green beans, and requiring either a move just involving only the white ones or involving only the green ones. It's really quite an interesting theory.

Oh, yes! There is a theory for the ***misère* play.** Now, this is when you want to lose: you play these games according to the convention that if you can't move, you win. My trouble with playing these games is that I suddenly realize halfway through the game, when I've been carefully setting up the position, that I've been playing in order to lose deliberately, or whatever it is that I wasn't supposed to be doing.

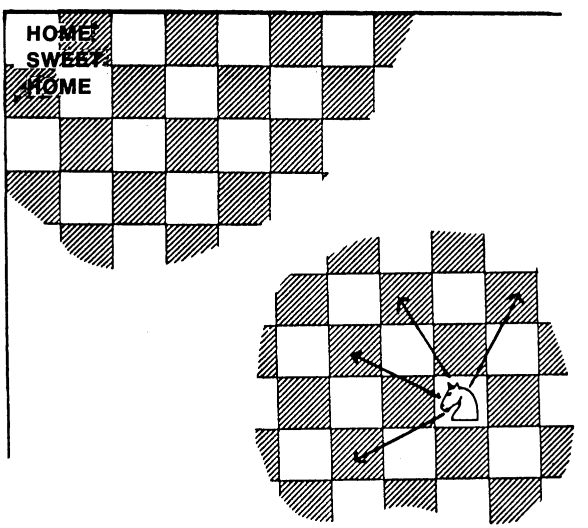
However, in many games — but not really in as many as all that — it turns out that it doesn't matter until very near the end of the game. In Nim, it's the case that if you want to play the losing game, the *misère* version, then you just play the winning game until you're in danger of leaving heaps of size 1 only; then you suddenly change so that you leave the wrong number. In the winning version, you try to leave an *even* number of heaps of size 1; in the losing version, you try to leave an *odd* number. It's thought by far too many people that that's what happens in general: that the losing version is just about the same as the winning version. But there are very few games, really, for which that is true — although there are some quite interesting ones among them. For the general game, the theory of *misère* play is very hard indeed.

Conjunctive compound games and the Steinhaus remoteness function

The next kind of game theory is the **conjunctive compound** — or the **join** of games for which we write $G \,\&\, H \,\&\ldots$. In that, the legal move is to move in *every* one of the components. So suppose you're playing Nim conjunctively. You've got a number of heaps. The move is to make a Nim move in every heap; in other words, to reduce the size of every heap by so much. Oh well, I can see how to play this, can't I? You've got a whole lot of heaps, and it's my go: well, I'll take them all. And you can't move, so you lose.

However, in some games, it gets much more interesting. Let me give you a little example. Suppose we have a checkerboard again. It does not very much matter how big it is this time; so I'm going to think of it as a quarter-infinite chessboard. And you have a number of things I'll just call "horses", that look like the knights in chess and move in the same way, except that they've always got to move vaguely nearer the corner. A horse on any square can move in one of the four directions shown in FIGURE 2a. Note that squares are supposed to be big enough to allow several horses at once.

When it's your move, what do you do? You've got to move every horse. Well, it's too bad, isn't it, if you happen to have any horse stuck in the 2×2 piece in the corner; because if you're stuck up there, there's no move the horse can make without jumping off the board, which is illegal. So let's delete that piece and call it "Home Sweet Home". Roughly speaking, the first person to get a horse "home" is the winner, because the next person is going to be in this awkward position of not being able to move. So *first home wins.* That's a corollary of the definition that if you can't move, you lose.



(a)

0	0	1	1	2	2	3	3	4	4	5	5
0	0	1	1	2	2	3	3	4	4	5	
1	1	1	1	3	3	3	3	5	5		
1	1	1	3	3	3	3	5	5			
2	2	3	3	4	4	5	5				
2	2	3	3	4	4	5					
3	3	3	3	5	5						
3	3	3	5	5							
4	4	5	5								
4	4	5									
5	5										
5											

(b)

The possible moves for a horse and some values of the remoteness function.

FIGURE 2.

What on earth do you do in a game like this? Well, this is really quite interesting, if you think about it. Suppose we've got dozens of horses scattered all over the board: which horses do you pay special attention to? Well, pretty obviously, the "favorites!" The horses that are very near the top lefthand corner have a very good chance of determining the whole game; because if a horse is a million miles away, it's very unlikely to be the first one home, and it's only the first one home that counts. So you've got to measure, in some sense, how remote the horses are from home; and then you pick the favorite, which is the one nearest home, and pay special attention to it.

These things are solved by a function called the **remoteness function,** which is due to Steinhaus. Imagine you're just playing a one-horse game, not a particularly fascinating game, but it's interesting enough. Then, what happens? Well, whoever thinks he's going to win is concerned to make this horse move as quickly as possible and win pretty quickly. Whoever thinks he's going to lose is going to drag it back. The game will take a certain length of time which is the number of moves it takes, supposing the winner wants to win quickly and the loser wants to lose slowly. This number of moves is the remoteness of the game.

Why is that a useful function? Suppose you've got a lot of horses. Then the ones that win for you are the ones that you want to get home. So if you can win with a particular horse, supposing that were the only horse in the race, then you should try to win very quickly, because it may be the critical horse that wins for you. If, on the other hand, you can see that the other player can win with "your" horse, then try and hold it off — keep it back, in the hope that a horse that you can win with can get to the winning post first, so that this "losing" horse is irrelevant.

The rule is that the remoteness of $G \& H \& \dots$ is the minimum of the remoteness of G and the remoteness of H and so on:

$$R(G \& H \& \dots) = \min(R(G), R(H), \dots).$$

You can work out remoteness very easily. I'll do it very quickly (in FIGURE 2b) just to show you what goes on, and only for a small interval. The home squares have remoteness 0. A horse in one of the squares bordering home can be sent home in just one way (and that's a good move!), so their remoteness is 1. Let's have a look at the square 2 across and 4 down; denote it $\langle 2,4\rangle$. It can be moved either to a square of remoteness 0 (home) or to one of remoteness 1 (namely, $\langle 3,2\rangle$). They're the only two legal moves from that square; the others take you off the board. Which one do you make? Well, if you make the first, it's a winning move; because if this is the only horse in the race, you make the move and then call upon your opponent to move, which he can't. But the second is a losing move: because if you do it, he will have to win.

So what happens is that *even* remotenesses correspond to good moves: you make a move, if you can, to a square of even remoteness. A horse at the square $\langle 5,2\rangle$ can only be moved to squares of remoteness 1; they're losing moves — too bad. A horse at $\langle 4,4\rangle$ can be moved to squares of remoteness 1 ($(\langle 3,2\rangle, \langle 2,3\rangle)$), but it can also be moved to squares of remoteness 2 ($(\langle 2,5\rangle, \langle 5,2\rangle)$). A move to a square of remoteness 2 is a winning move, so it's rather better. If you've got a choice of a lot of numbers to move to, move to an even one if you can; of several even ones, you move to the least one, because that corresponds to winning quickly. If on the other hand, you've only got odd numbers, what do you do? You move to the greatest ones because that corresponds to losing slowly. And then, if you've got a lot of horses, scattered around the board, you choose the one of minimal remoteness, which decides who wins. You look to see whether it's odd or even and that tells you whether you have a winning move or not.

Continued conjunctive compounds and the suspense function

Now suppose we change the game a bit and say that whenever a horse is home, it is removed. You can think or it like this: a horse that's home can't be moved, so there must be something wrong with it; so we'll take it off the board — take it to the vet or something and have a look at it. And so the number of horses in the race gradually decreases, until the *last* horse is the one that controls the game.

Whoever makes the last move with that is really the winner, because the next person sees no horses at all on the board and therefore can't move.

Now what happens in that game? In that game it's the last component that determines the game. It's called a **continued conjunctive compound.** It's one where you don't have to move in *every* component, just in every component where you *have* a legal move. In the continued conjunctive compound, it's the *last* horse in the race that determines the end of the game.

The old technique of winning games (if you can) quickly and losing slowly is not really as enjoyable as another technique where, if you win, you want to drag it out as long as possible and watch your opponent squirm away; and if you lose, rather than having this horrible business of playing for three more years knowing you're going to lose, you'd like to get it over with quickly. And that gives rise to the **suspense function** of G. It's computed in a manner similar to that used for the remoteness function, except that if you're choosing between several even numbers, you choose the *largest* even number. That corresponds to winning, so you want to win slowly. If you're choosing between a number of odd ones, you choose the *smallest* one, because you're losing. It's still true that if you've got some evens and some odds, you choose an even, because you do prefer to win rather than lose. The suspense function of a compound played according to these rules is the *maximum* suspense function of the components because it's the last horse in the race that determines the winner — the outsider.

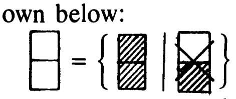
Theory of selective compounds

Now I'm going to illustrate the selective compound theory with just one game, which we call "Frosted Cakes". It's a game deliberately invented to be rather special. Well, what happens? Mother's just made these things, and I think they're called "brownies" or something over here. They're scored into little squares to be broken up and eaten. Lefty and Rita are the two children. Lefty eats a full vertical strip through a cake, in fact, through each of as many cakes as he likes. But if you see a 1×1 square, eat it and win outright!

What goes on? Let's start with a 4×4 cake. Lefty eats away one of the two vertical strips through the middle; that vertical strip has now disappeared, so there are now two cakes on the board. So Rita, if she likes, can make moves in both of them. Well, let her! You can eat strips from as many cakes as you want to — this is what is called the **selective compound.**

What on earth do you do in this game? It turns out that it brings out what is really one of the most interesting theories. If you see a 1×1 cake, it means that somebody's just about to win or something, so we needn't delay further with the analysis.

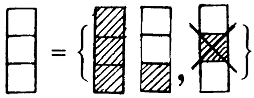
What happens if you see a 2×1 cake, vertically arranged? What should you do? Well, I'm now going to write the kind of notation that is customary in this sort of theory. Lefty has no option, if that's the only cake that he sees, but to eat a vertical strip right through it. Rita has no option but to eat a horizontal strip, which consists of just one square. This is really rather sad for her, because if she eats that horizontal strip Lefty will win on the next move. So she doesn't want to do that — a bad move! But if that's the only cake on the board, then "hard cheese!" (or "hard cakes!"). So this is really a sort of infinitely bad move for her and you can delete it. You might as well make illegal all moves that lose instantly, because it doesn't affect the outcome. This is a move to 0 for Lefty. Rita really has no move. We display this in the manner shown below:

 $= \{0 \mid \ \}$

Now there's a theory of numbers, due to somebody too modest to name himself, according to which the value of this game is the simplest number greater than 0: $\{0 \mid \ \} = 1$. That really means this game counts as 1 move for Left; -1 would be 1 move for Right, so that

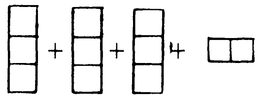
 $= \{ \ \mid 0\} = -1.$

Let's look at what happens if you choose a 3×1 strip. What can Left do? Well, he's got no option but to eat it all. Right? Well, she's got an illegal move, we might say, of cutting through the middle, which loses instantly. She's also got the move of eating an end square. What's the value of the remainder on the left side? 0. The value of the remainder on the right is 1, so the value of the position is



$$= \{0 \mid 1\} = \tfrac{1}{2},$$

because according to this wonderful theory, the "simplest" number greater than 0 and less than 1 is 1/2. So that position counts as a half a move advantage for Left. Now, that's really rather nice, if you have on the board, say, three copies of the 3×1 and a 1×2, and those are *all* the cakes left on the board, then you see that the value is



$$= 1/2 + 1/2 + 1/2 - 1 = 1/2.$$

It is the content of this theory that this process works. Left is half a move ahead and should win. If the number adds up to a positive number, Left wins; if it's a negative number, Right wins; and if it's zero, then whoever doesn't start wins, because zero is really the empty situation, in which you can't move, so you lose.

Of course, you get much more interesting games if you take, say, a 5×11 cake; then the value turns out to be 61/128. In other words, from Left's point of view, this is worth 61/128ths of a move to him. Other games, say a 5×7 cake, do not yield numbers for their values. Both players can improve their position by moving, so they do. Eventually the value of the game will become a number. When that happens, as it did, say, when it was 1/2, nobody wants to move in it: it's a terribly cold position! Left is half a move ahead, but if he makes a move, he'll only be 0 moves ahead, and he's lost half a move. And again, Right doesn't want to move, because if Right moves, then Left will be 1 move ahead, which is even worse. So this is what's called **cold.** It's a property of numbers in general that they're cold — you don't want to move.

So what happens in these games is that you have what we call a "hot battle". You've got a whole lot of cakes all over the place, and some of them you want to move in: they're **hot.** So you move in all the hot ones, like mad; and then eventually they get cold. And as each game gets cold, nobody wants to move in it, so it stays static. The hot battle gradually dies out, so when all the hot things have gone you have what we call the "cold war", in which nobody wants to move anywhere; so they make the least disadvantageous move.

Well now, how do you play such a thing? This is the most fascinating thing of all, really! You want to force your opponent to make the first move in the cold war. So, if you can win anything, you want to drag it out as far as possible; if you must lose, you want to make it end quickly.

That's where suspense functions come in. If you take a 6×17 cake, then you find it's a truly wonderful thing. The value of this game is:

$$\frac{133}{128_3}, \quad \frac{2039}{2048_5}.$$

Here the fractions are the ultimate values, and subscripts the suspense functions according as Left or Right starts. What the subscript 3 means is that if Left starts, the game will take exactly 3 moves before it gets cold. These 3 moves will just be made clunk! clunk! clunk! Everybody's keen to move in this game at the moment; it's hot! They play exactly 3 moves before it gets cold, and when it gets cold, the value will be a number, which is 133/128. If, on the other hand, Right starts to move, it's going to take 5 moves before it gets cold. And then when it gets cold, the value will be a different number, which is 2039/2048. Because the first number is visibly bigger than the second one, the game is hot: Left by moving gets a bigger number, more in favor of him, than Right does. Very small fractions are involved here because, as you see, 133/128 on the left is just a little bit bigger than 1, while on the right 2039/2048 is just a little bit less than 1. So the difference isn't all that much. But you do want to move in there, just to pick up that extra fraction.

It's a very funny thing, this game. Just look at the table of values and you'll find that they're completely chaotic:

Game size:	5×7	5×8	5×9	5×10	5×11	5×12
Value:	$1_2,\ -1/32_2$	0	1/4	3/8	61/128	$1/2_1,\ 503/1024_3$
	Hot	Cold				Hot

If you see a 5×8, 5×9, 5×10, or 5×11 cake, you don't want to move in it, oh no! If you see a 5×12 cake, oh yes! you must move! If you see a 5×7 cake, you've got to move as well: I mean, you stand to lose a tremendous amount if you don't move in a 5×7 cake, because Left, if he moves, is one move ahead; Right, if she moves, is 1/32 of a move ahead. So there's 33/32nds of a move difference in there, and you want to move like hell in a 5×7 cake.

So roughly speaking, this is what goes on: you have these wonderful suspense functions, which tell you how long it's going to take to get cold, and then they determine who has the move in the cold part; moving in the cold part is disadvantageous, so therefore you don't want to do it.

The theories can be applied to hundreds and thousands of games — really lovely little things; you can invent more and more and more of them. It's especially delightful when you find a game that somebody's already considered and possibly not made much headway with, and you find you can just turn on one of these automatic theories and work out the value of something and say, "Ah! Right is 47/64ths of a move ahead, and so she wins".

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Three Annihilation Games

A new class of superficially simple games in which tokens move to adjacent squares and are removed when two tokens cohabit.

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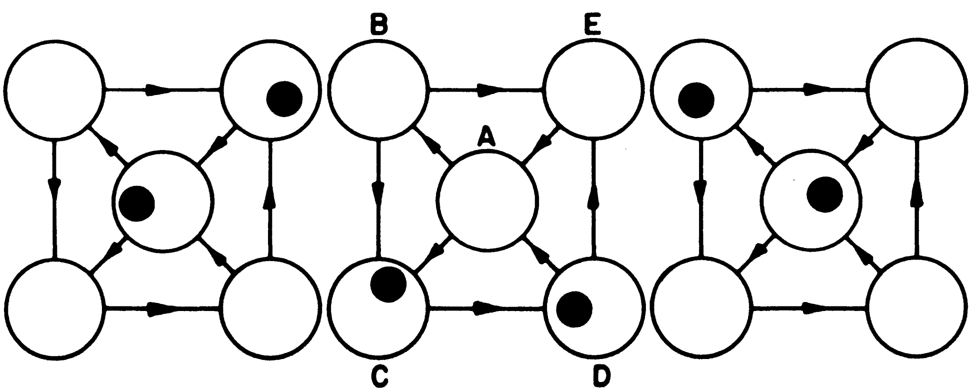
Gill and Jean were very bright high school students. They successfully survived a large number of "New Math", "New Physics", "New Biology", ... programs. They succeeded, in fact, in aquiring a good general education despite the onslaught of all the new programs and the old teachers. They were, of course, exceptionally talented.

G: (Drawing in the sand with his hands) Hey, Jean, I just invented a new game of marbles (Figure 1).



Gill's original suggestion for playing "Innocent Marbles" on three identical graphs.

Figure 1.

J: (In a bored voice) Really? What is it?

G: Place a number of marbles into any distinct pits of the three identical graphs that I drew. Two players play alternately, each player at his turn selecting any marble and moving it into a neighboring pit along an edge in the direction of its arrow. If the pit the marble moves into is occupied by a marble, both marbles are transferred into a special pot. The player making the last move, i.e., removing the last pair of marbles from a pit into the pot, gets the pot and is the winner.

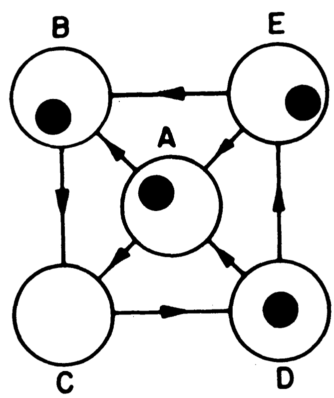
J: But if any number of marbles may be placed into distinct pits initially, I could place one or three or five in any one of the graphs, and then the game is clearly a tie, because there will be a marble that can never be taken into the pot.

G: Hm... yes. You got a point there. So in order to make the game interesting, an even number of marbles ought to be placed on each graph.

J: Incidentally, I think that there may be ties also if an even number of marbles is placed on each graph, because of the cycles, such as *ACD*, that the graphs contain.

G: Possibly. I suggest that we find out by playing the game with the given initial configuration of just two marbles in pits *AE*, *CD* and *AB* in the three graphs. Please accept my offer to make the first move.

J: (Retrieving the portable console from her schoolbag which communicates with the school's computer by remote control, swiftly inputting some data and listening intently to the computer's advice) I accept your gentlemanly gesture offering me the first move with a bow. But a game on three identical graphs may be rather boring. Allow me therefore to add (drawing in the sand with her fingers) the following graph which is identical to your graphs, except that the direction of the edge *BE* is reversed (FIGURE 2). To make the game still more interesting, I put marbles into the four pits *A*, *B*, *E* and *D* in this new graph.

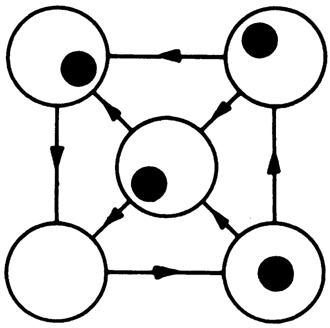


Jean's addition of a slightly modified graph.

FIGURE 2.

G: (Pulling his portable console from the hindpocket of his jeans, consulting it frantically and then drawing in the sand.) In order for you to realize that the game is not boring even with identical components, I shall add a graph identical to the one you added, with the marbles placed exactly in the same pits (FIGURE 3).

Jean now agrees to play on the five graphs drawn in the sand, since she knows she will not lose. The configuration originally laid out by Gill was such that whoever moves first loses. A gentlemanly gesture indeed! By adding her graph, Jean changed the situation fundamentally. The player now moving first can win (either by moving from *A* to *C* in the graph she added, or from *E* to *A* — with annihilation). Realizing this, Gill added a graph to make the game a (dynamic) tie. That is, neither player can now force a win; therefore both can avoid losing, except that if one of them commits a strategy error his opponent can win. Incidentally, instead of adding a fifth graph, Gill could have achieved a tie by adding a pair of marbles on *AB* or *AE* in the middle graph of FIGURE 1.

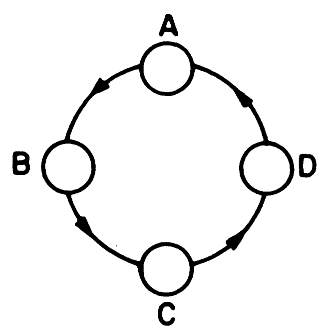


Gill's addition, identical with Jean's addition.

FIGURE 3.

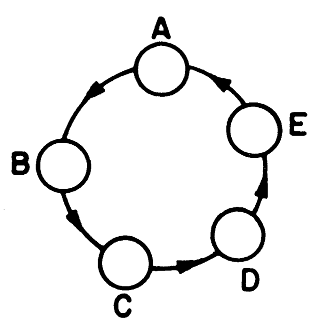
Gill and Jean's game, named the Innocent Marble Game, is an example of a new and large class of board games collectively termed annihilation games. They were originally suggested by John Conway (private communication). In these games a number of pieces, all of the same color and shape, are placed on a subset of distinct vertices of a finite directed graph R without loops. Each player at his turn selects one piece and moves it to a neighboring vertex along an edge in the direction of its arrow, say from vertex u to vertex v. If there was already a piece at v, both pieces get annihilated, i.e., they are taken off the graph and out of the game. The player making the last move is the winner. If there is no last move, the game is a (dynamic) tie.

Annihilition games are non-trivial only if R contains cycles: If R contains no cycles, there exists a unique classical Sprague–Grundy function g on R [**1**] which completely determines the strategy, analogously to the binary number strategy for Nim. Since $g(u)\oplus g(u)=0$ (where $\oplus$ denotes Nim-addition), the game in which two pieces may coexist on a vertex u is equivalent to the annihilation version, where two such pieces vanish. The situation is essentially different if R contains cycles (where g may not exist uniquely). For example, suppose that two pieces are placed on vertices A and C in FIGURE 4. Then the second player can win, since the first necessarily moves to (B, C) or (D, A), and so the second player can annihilate. If no annihilation takes place, the outcome is clearly a (dynamic) tie. But the position (A, C) in FIGURE 5 is seen to be a (dynamic) tie position, in both the annihilation and non-annihilation version.



Two pieces on vertices A and C are a second player win in the annihilation game, but a tie in the non-annihilation version.

FIGURE 4.



Here, two pieces on vertices A and C are a tie in both the annihilation and the non-annihilation version.

FIGURE 5.

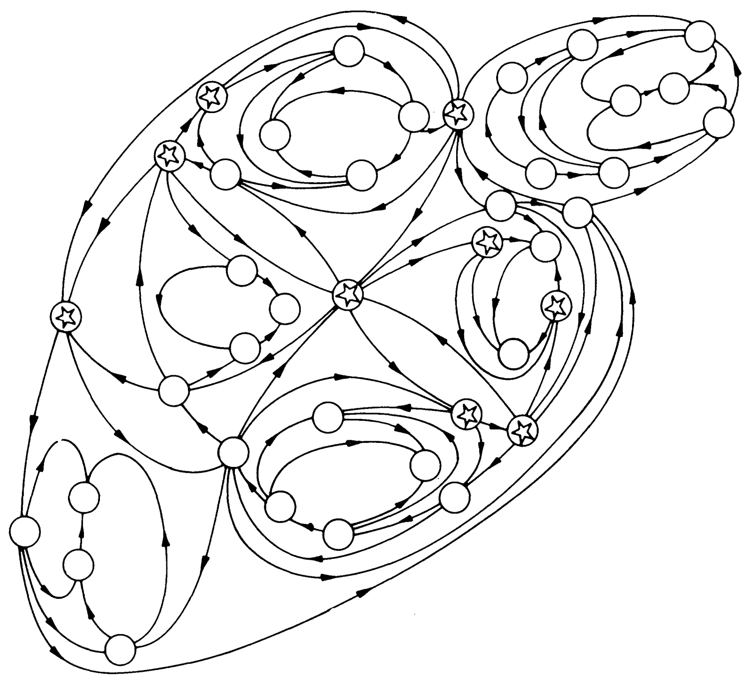


FIGURE 6.

Here is a more dynamic illustration of annihilation combined with dynamic ties — the game called Worlds in Collision. "Stars" or "worlds" are placed on distinct "space stations" of the given "universe" (see FIGURE 6). Two super spacemen play alternately. Each player at his turn selects a star and moves it to a neighboring space station along an edge in the direction indicated by its arrow. If the space station it moves into is occupied, the colliding stars explode and vanish in the universe in a cloud of interstellar dust. If it moves into a space station without an exit (there are six such stations in the universe), it phases out of the universe ("falling star"). Otherwise it just stays in its new station waiting to be moved again or exploded by an incoming star.

Consistent with the new "code" of brute violence threatening to destroy mankind, the player who knocks off the last star(s) is the winner. There might be a situation where there is no last star being knocked off. In this case the game is a tie, and at least one star continues to streak peacefully through hostile space.

Unscrupulously destructive minds may indeed believe that they can knock off the last world(s). Most of mankind may even by scared into entertaining such beliefs. But the truth is that the world destroyers can succeed to destroy some worlds perhaps. Destroying too many worlds inevitably leads to their ultimately losing — the game. If you do not believe it — join the Goliaths and other forces of evil. This will force David to explain the facts to you. For example, the configuration of the sun's 9 planets (marked ☆ in FIGURE 6) is a tie. If a player annihilates too many worlds, or the wrong ones (e.g., the two "easternmost" worlds in FIGURE 6), he forfeits the ability to force a tie, and his opponent can win.

We conclude this introduction to annihilation games with a more explicitly mathematical game known as the Battle of Numbers. It is played on a (finite) linear board consisting of squares numbered consecutively $0,1,2,\ldots,n$. Let $0<2b\leqq a<c$ (a,b,c integers). Two players play alternately with a number of pieces placed on distinct squares of the board. Each player at his turn selects a piece, say on square m, and makes one of the following two possible moves:

(i) Move the piece to square $m-k$, where $1\leqq k\leqq a$ or $c+1\leqq k\leqq c+a$.

(ii) If $m\equiv 2b \pmod{2a}$, move the piece to $m+a$.

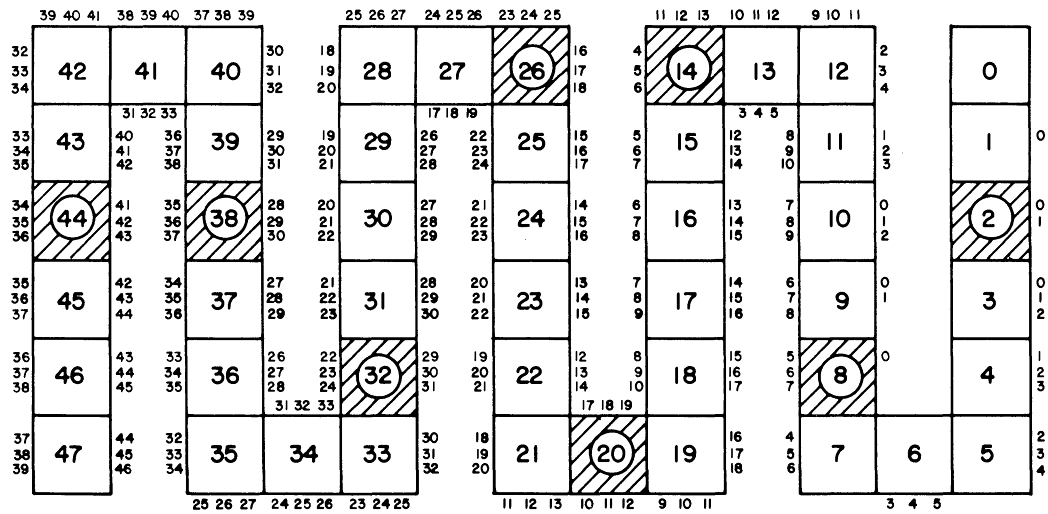


FIGURE 7.

In either of these moves, if a piece lands on an already occupied square, both pieces get annihilated. The player making the last move wins. If there is no last move, the game is a (dynamic) tie.

For example, suppose $n = 47$, $a = 3$, $b = 1$ and $c = 7$ (see FIGURE 7) and seven pieces are on squares 13, 19, 21, 26, 36, 37, 45. (For convenience, the squares one can move to from any given square are indicated by the numbers bordering the squares, except that, in addition, pieces on squares $2, 8, 14, 20, \ldots$ can move (by rule (ii)) to $5, 11, 17, 23, \ldots$; these special squares are shaded in FIGURE 7). If you move first, you can win, even though one of the pieces is on square 13! But your only chance is to move from 45 to 36 and annihilate. (The same seven piece configuration is a tie if no annihilation takes place, i.e., if two or more pieces can coexist peacefully on every square.)

A non-annihilation version of the Battle of Numbers game was first proposed in [**3**], where its winning, losing and tie positions were determined for the case of a single piece. The strategy for any number of pieces was determined in [**4**], and the annihilation version was solved in [**7**, Part III]. The general theory of non-annihilation games with possible (dynamic) ties is explored in [**1** (Chap. 11), **3**, **5**, **8**]. (It is based on a generalized Sprague–Grundy function G, which can assume the values $0, 1, 2, \ldots$ and ∞.) The general theory of annihilation games is developed in [**2**, **6**, **7**].

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What is Recreational Mathematics?

Definition by example: paradigms of topics, people and publications.

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Mindful of the way Tom Sawyer got his fence whitewashed, I asked a number of mathematicians (including the speakers at the Miami University Recreational Mathematics Conference) to answer the question, "What is recreational mathematics?" The response was gratifying and illuminating. Many will recognize their brushmarks in the following discussion. For those who detect untouched areas on the fence, there are plenty of brushes available.

One thing is immediately clear: defining "recreational mathematics" is not recreational. The difficult task of defining "mathematics" is not simplified by the qualifying "recreational".

"Recreation" is defined in **[1]** as "a pastime, diversion, exercise, or other resource affording relaxation and enjoyment". One indulges in recreation to re-create oneself, to relax from work-a-day pursuits, and to clear and refresh the mind before returning it to its regular occupation. The closely related word "play" is defined in **[1]** as "exercise or action by way of amusement or recreation". Leonard **[2]** asserts that "all creative activity begins with play, to which will is then applied, as much in science and art as in sports. The Pythagorean theorem and the model of the double helix are at least as proportional, fit, elegant and admirable as the one-handed jump shot and the perfectly executed double play. Moreover, they are fun". Indeed, W. F. White **[3]** once observed that "amusement is one of the fields of applied mathematics".

"Mathematical work is highly satisfying. So is mathematical play. And, as most often is the case, one is apt to work much harder at any form of play, mental or physical, than one would for mere remuneration. Mathematical activity, more than any other, gives scope for the exercise of that faculty which has elevated man above other creatures" **[4]**.

It is human nature to enjoy those things that can be done well. It is also human to resist mandatory tasks. At the University of Michigan the late Norman Anning used to take advantage of this human frailty by offering an intriguing non-credit problem at the end of each class section. The problem may or may not have been related to the course content, but it stuck in the long-time memories of many of his students. Some rather dry sounding problems can be recast into very nice recreational settings.

"Mathematics can provide enjoyment for a variety of reasons — meeting the needs of those who seek recreation, while giving satisfaction to those who are intrigued by solving problems, close reasoning and unexpected solutions" **[5]**.

The allure of problem solving may be a matter of ego satisfaction. The joy of triumph over an opponent can be equalled or surpassed by the glow of finding a solution to a tough mathematics problem, even though it may turn out that the result had been published in an obscure 19th century journal. That many enjoy such challenges is attested to by the popularity of problem sections in various magazines. There the student who works all the problems in the textbooks just for the joy of it can further test his mathematical mettle. Decades ago Richard Bellman, by then an accomplished mathematician, told me that he had cut his mathematical eye-teeth on the challenge problems in *School Science and Mathematics.*

Challenge problems also can widen the mathematical horizons of those whose mathematical development has not been blocked by unfortunate circumstances, even though they do not follow mathematics professionally. They have the tools to pursue an inexpensive and stimulating recreational activity. Their laboratory and shop consist of pencil and paper. At times, this recreation may re-create the work of others, but the possibility of discovering new relationships is always present. Howard Eves once compiled a long list of original articles that had been inspired by the problem departments of the *American Mathematical Monthly.*

Mathematics affords an ever-new never-boring avocation both before and after retirement where it is an effective weapon against vegetating. The human interest section of the Otto Dunkel Memorial Problem Book [**6**] lists the names of a nurseryman, a lawyer, a ballistics expert, a dentist, a steel works manager, a retired telephone engineer, an automobile dealer, an insurance inspector, a patent examiner, and a clergyman who were contributors to the problem departments of the *Monthly.* It is of more than passing interest that the problem departments of the journals of the two college level mathematical fraternities (Pi Mu Epsilon and Kappa Mu Epsilon) are edited by a practicing dentist and a practicing attorney — Dr. Leon Bankoff in the *Pi Mu Epsilon Journal* and Kenneth Wilke in *The Pentagon.*

In the less austere days of the problem departments of the *Monthly*, the late Norman Anning proposed the problem [7] to "find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D, in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}$$

The published solution remarked that the value, $(1-a^4)^3$, of the determinant was independent of the values of B, C, and D, so their insertion merely changes the appearance of the determinant and not its value. "Thus, the element of likeness in (a), (b), (c), (d), and (e) is that only the appearance of the principal entity is changed. The same element appears also in: (f) changing the name-label of a rose, (g) changing a decimal integer to the scale of 12, (h) gilding a lily, (i) whitewashing a politician, and (j) granting an honorary degree". Anning sent the solver a cartoon, clipped from the Saturday Review, of an Indian watching the cloud of an atomic blast. The caption: "I wish I had said that".

Other irreverent contributors to the problem departments are those with risible nom-de-plumes, such as ALICE MALICE, POLLY TOPE, NEV R. MIND, ZAZU KATZ, and ALFRED E. NEUMANN of MU ALPHA DELTA FRATERNITY. My favorites are NOSMO KING and BARBARA SEVILLE.

Still other recreational support of the tenet that mathematics is too important to take too seriously are: the sporadic appearances of Professor Euclide Paracelso Bombasto Umbugio, the priceless lyrics of Tom Lehrer, Leo Moser's verse [**8, 9**], the Mathematical Swifties ("The angle is less than 90°, Tom noted acutely") in the 1964 *American Mathematical Monthly*, the Show Me's ("Show me a man who counts on his fingers and I'll show you a digital computer") in the *Journal of Recreational Mathematics*, and the varied offerings in that delightful new Canadian periodical, *Eureka* [**10**]. The disdainful may say that in mathematics a little humor goes the wrong way.

Few will disagree with the classification of mathematical humor, poems, anagrams, rebuses, word equations, cross-number puzzles, acrostics, and cryptarithms as purely recreational, although they have some educational use. Some may question whether they qualify as mathematics. However, to exclude the anecdotes of Eves' *In Mathematical Circles* would be to exclude a portion of history of mathematics as well. But what of other topics?

In the preamble to his excellent discussion of "Number Games and Other Mathematical Recreations", [**11**] William L. Schaaf remarks, "Mathematical recreations comprise puzzles and games that vary from naive amusements to sophisticated problems, some of which have never been solved. They may involve arithmetic, algebra, geometry, theory of numbers, graph theory, topology, matrices, group theory, combinatorics (dealing with problems of arrangements or designs), set theory, symbolic logic, or probability theory. Any attempt to classify this colourful assortment of material is at best arbitrary". However, in his *Bibliography of Recreational Mathematics,* [12] Schaaf does impose a classification by means of the chapter headings: arithmetical recreations, number theory as recreation, geometric recreations, topological recreations, magic squares and related configurations, Pythagorean recreations, recreations in antiquity, combinatorial recreations, manipulative recreations, miscellaneous recreations, mathematics in related fields, and recreations in the classroom.

Most of the mathematics books with "Recreational", "Play", "Amusement", "Fun", "Diversions", "Pleasure", "Entertainment", or "Pastimes" in their titles are problem oriented. Predominant among the topics dealt with in such books are cryptarithms, magic squares, dissections and tessellations, decanting liquids, measuring, weighing, packing, shortest paths, calendars, the census, sliding movement, chess, dominoes, cards, river crossing, match arrangements, networks, permutations, combinations, diophantine equations, number properties, and various games.

In his detailed discussion of the first recreational mathematics book, that written by Bachet **[13]**, Dudley **[14]** remarked that "problems of a recreational nature appear on Babylonian tablets 3500 years old" and "the Egyptian Rhind papyrus of about 1650 B.C. contains" a "problem that must have been made up for the fun of it". Dudley also reports the existence of other recreational problems in various works of the 16th century and earlier.

Those interested in the evolution of recreational mathematics through the ages will find an excellent historical account in **[11]**, and a very good treatment of an international cross-section of recreational books in the Postscript of O'Bierne's *Puzzles and Paradoxes* **[15]**. The classic four volumes of Lucas **[16]** now appear in a French paperback edition. English translations are available of the recreational works of the Russians Domoryad **[17]** and Kordemsky **[18]**, the Polish Steinhaus **[19]**, the Dutch Schuh **[20]**, and the Belgian Kraitchik **[21]**.

Kraitchik was the editor of the defunct *Sphinx* (1931–1939), a magazine devoted to recreational mathematics. Its American counterpart, *Recreational Mathematics Magazine* (1961–1964), was also short-lived, although its successor, *Journal of Recreational Mathematics* (1968–), is flourishing.

The contemporaries Dudeney in England and Loyd in the United States were prolific inventors of mathematical puzzles which appeared in various periodicals. They were great rivals and did not hesitate to borrow from each other. Some of their efforts are preserved in **[22, 23, 24]**. Somewhat later Hubert Phillips under the *nom de plume* Caliban contributed problems to English periodicals, some of which are collected in **[25]**.

One of the great stimuli of interest in recreational mathematics is Ball's comprehensive book **[26]** now in its 12th edition. Another great stimulus is the *Scientific American* column "Mathematical Games" written by the dean of contemporary recreational mathematics, Martin Gardner. A list, by title, of his skillfully written columns, beginning with "Flexagons" in December 1956, appears in **[12]**. Much of the column material has been assembled in his many books, such as **[27, 28]**. For a multitude of other worthy volumes see **[12]**.

Some consider that for mathematics to be recreational there must be some element of play or games involved. But many who abhor games get great pleasure from other branches of mathematics. It may be a matter of definition of "games"; and if we accept Goodman's statement **[29]** that "Mathematics is the greatest game ever invented by man," we are back where we started from in search of a definition.

For many individuals, as they approach the limit of their abilities, mathematics loses its fun aspect. When a topic is undeveloped, it is recreational to many. As the theory is developed and becomes more abstract, fewer persons find it recreational. Of course, there are some who get their pleasure by concentrating on a single topic, pursuing it to the extent of current knowledge, and then trying to add something new. In many cases, what starts out to be purely recreational develops into an extensive discipline of serious mathematics such as now exists in number theory, topology, combinatorics, and game theory.

Many consider mathematics recreational if it is sufficiently elementary to be understandable by the non-mathematician. Recreational mathematics is "something you can explain to a business man sitting next to you on a flight from Chicago to Cincinnati". It is "a piece of mathematics that is both subtle enough to interest the professional mathematician and simple enough to be accessible to the man-in-the-street".

Others consider recreational mathematics from more of an academic slant: it is "a mathematician's holiday" that falls outside the bounds of standard school and university mathematics. "Learning what others have done is not recreational, doing it yourself is". Recreational mathematics is that which one works on "without thought of practicality, generality or academic rewards". "A recreational topic loses its standing when one squeezes from it an article to enhance the publication list."

Still others approach the topic from the standpoint of personal enjoyment. Surely the great amateurs Omar Khayyam, Leonardo Da Vinci, Blaise Pascal, and Pierre Fermat considered all mathematics to be fun. Indeed, many professional mathematicians consider all mathematics to be pleasurable. Richard Guy has said, "It always fascinates me that people are willing to pay me for doing what I would do for enjoyment in any case."

Recreational tastes are highly individualized, so no classification of particular mathematical topics as recreational or not is likely to gain universal acceptance.

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The Mystique of Repunits

A survey of prime factorizations, relations to periodicity of decimals, and other aspects of numbers which consist of a repeated unit digit.

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The Random House Dictionary defines *mystique* as a framework of doctrines, ideas, beliefs or the like, constructed around a person or object, endowing him or it with enhanced value or profound meaning. That undoubtedly describes the attributes of repunit study much better than the word *theory*, which is defined only as a body of principles, theorems or the like, belonging to one subject. The word repunit is not yet defined in any dictionary, having been introduced relatively recently by Professor A.H. Beiler of Brooklyn [**1**, p. 83] and having had relatively little (but nevertheless gradually increasing) exposure.

A **repunit** R_n is an integer written as a string of n ones such as 11 or 111. Its name comes from the fact that it indicates a repetition of the unit symbol. Unless otherwise stated, we will assume that our numbers are expressed as numerals with ordinary base 10 positional notation. However, it should be noted that other bases can be considered as well. In a number system whose base is b, b^n is written as a 1 followed by n 0's. For instance, in base 10, 10^3 is 1 followed by 3 0's; and in base 8, 8^3 appears as a 1 followed by 3 0's. It follows that $b^n - 1$ in base b notation is written as a string of n $b-1$ symbols. For instance, $10^3 - 1$ in base 10 is 999, and $8^3 - 1$ in base 8 is 777. It is clear then that $b^n - 1$ divided by $b - 1$ is a string of n 1's, a repunit expressed in base b. For $b = 10$, it follows that for each natural number n, $9R_n = 10^n - 1$.

Part of the fascination of repunits to those who enjoy mathematical recreations is their close relationship to many mathematical curiosities. Consider, for example, the product of R_3 and R_5 as it is carried out below:

$$\begin{array}{ccccccc} & & 1 & 1 & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & & \\ \hline 1 & 2 & 3 & 3 & 3 & 2 & 1 \end{array}$$

The resulting product is an example of a numerical **palindrome,** a number which is the same whether it is read from left to right or from right to left. The computations in this product make it clear that if the repunit R_i is multiplied by the repunit R_j, where j is less than or equal to both i and 9, then the resulting product will be a palindrome of $i + j - 1$ digits, the first j digits of which are 1 through j, read either way. The number of j's that appear in the middle is $(i+j-1)-2(j-1) = i-j+1$. For example,

$$R_9 \cdot R_8 = 1234567887654321$$

$$R_{13} \cdot R_3 = 123333333333321$$

$$R_{100} \cdot R_4 = 123444 \cdots 444321 \qquad (97\ 4\text{'s}).$$

Further discussion of palindromes can be found in [**2**].

Primes and Repunits

The most fascinating aspect of repunits, however, is the nature and curious properties of their prime factorizations. Part of the motivation for this investigation comes from the situation of repunits expressed in base 2. When the base b is 2 and n is prime, the repunit is known as a **Mersenne number,** named after the seventeenth century Frenchman who studied such numbers. If the repunit itself is prime, it is called a **Mersenne prime.** Twenty-four such primes are known today [**3**]. The largest is $2^{19937}-1$, which is written in the binary system as a string of 19,937 1's, but is a 6,002 digit number in the common decimal system. Large Mersenne numbers lend themselves better to current primality investigative procedures than do other large numbers, which is the main reason why our largest known primes are Mersenne primes. Aside from the 24 that are known there are no Mersenne primes less than $2^{19961}-1$ [**4**].

The distribution of primes among decimal repunits is even more sparse: there are only four of them less than R_{1031}, namely R_2, R_{19}, R_{23}, and R_{317}. However, the search for primes among these numbers has revealed some interesting patterns.

A number that divides R_n and no smaller repunit is called a **primitive divisor** of R_n. The remaining divisors of R_n are called **algebraic divisors** of R_n. The **primitive cofactor** of a repunit is the product of all of its primitive prime divisors, and the **algebraic cofactor** is the quotient obtained by dividing the repunit by its primitive cofactor. Table 1 gives the factorizations of the first few decimal repunits.

Repunit	Prime Factorization	Primitive Prime Divisors	Algebraic Prime Divisors	Primitive Cofactor	Algebraic Cofactor
R_1	1	1	1	1	1
R_2	11	11	1	11	1
R_3	3·37	3,37	1	111	1
R_4	11·101	101	11	101	11
R_5	41·271	41,271	1	11111	1
R_6	3·7·11·13·37	7,13	3,11,37	91	1221
R_7	239·4649	239,4649	1	1111111	1
R_8	11·101·73·137	73,137	11,101	10001	1111
R_9	3·3·37·333667	333667	3,37	333667	333
R_{10}	11·41·271·9091	9091	11,41,271	9091	122221

TABLE 1.

Extending this table becomes more difficult with increasing n. Although almost one fourth of the repunits up to and including R_{100} were completely factored a century ago, some of the rest of them have not yet been done. The first few complete factorizations of repunits over R_{100} began to be reported in 1976.

Both improved techniques and continually improving computers contribute to progress in factorization. Nevertheless, we still don't have even one prime factor of repunits such as R_{71}, R_{73}, and R_{101}, all of which have been shown to be composite. In general, the algebraic prime divisors of a repunit can be found before its primitive divisors because algebraic divisors are primitive divisors of smaller repunits. However, the reverse occasionally happens: until recently, for instance, only the primitive divisors of R_{98} were known.

When n is composite, $b^n - 1$ can be factored algebraically, giving among its factors $b^m - 1$ for all m which divide n. Consequently, all the prime divisors of R_m are algebraic prime divisors of R_n. We can see this illustrated in Table 1: the algebraic prime divisors of R_n where n is composite include those of R_m for all m which divide n. In fact, the table suggests that all the algebraic divisors of R_n are found in this way. For example, the algebraic prime divisors of R_{10} are precisely the prime divisors of R_2 and R_5; those of R_6 come from R_2 and R_3; those of R_9 come from R_3, etc. In general, an algebraic prime divisor of R_n must be a primitive prime divisor of R_m for some m less than n. We shall see later that m must indeed be a divisor of n. This means, for example, that if n is a prime, all prime divisors of R_n are primitive. Thus the algebraic cofactors of R_3, R_5, and R_7 are each 1. These facts make it possible to set up procedures for finding the algebraic and primitive cofactors of a repunit even when no prime divisors are known, and even to estimate the range of the number of primitive divisors **[5]**.

We have seen that the algebraic cofactor of R_n will equal 1 if and only if n is prime. A remarkable property of decimal repunits is that the primitive cofactor is never equal to 1. More specifically, it can be shown that, except when $b = 2$ and $n = 6$, every repunit in every base system has at least one primitive prime divisor **[6]**. Thus, each higher repunit is either prime or it is divisible by at least one prime which does not divide a smaller repunit. This reminds us of the classic proof that there is an infinity of primes, in which it is shown that for any n, because $A = n! + 1$ is not divisible by any $m \leqq n$, either A is a prime or A is divisible by a prime greater than n.

Another fascinating property of decimal repunits is that every prime number with the exception of 2 and 5 is a primitive prime divisor of some decimal repunit. One way of seeing this is to invoke the famous theorem due to Fermat which states that if n is prime and b is not a multiple of n, then n divides $b^{n-1} - 1$. For the case in which $b = 10$ and n is a prime other than 2 or 5, we then know that n divides $10^{n-1} - 1$. If, furthermore, $n \neq 3$, we can conclude (since $9R_{n-1} = 10^{n-1} - 1$) that n divides R_{n-1}, and therefore it is a prime divisor of R_{n-1} and a primitive prime divisor of some, possibly smaller, repunit. For the case $n = 3$, simply observe that 3 is a primitive prime divisor of R_3.

Periodicity

Repunits are closely related to periodicity and period lengths of primes. In order to make these ideas precise, consider the computation of the decimal equivalent of the fraction 1/7, which comes out to be .142857142857142857..., commonly called a **repeating decimal** and frequently indicated as $.\overline{142857}$, showing a bar above the repeating set of digits called the **period** or **repetend** of 1/7. The repetition occurs because in the partial division of 1 by 7, performed as a long division as shown here,

```
     .142857
  7 | 1.0000000
      7
      30
      28
       20
       14
        60
        56
         40
         35
          50
          49
           10
```

the successive remainders, as underlined, are $1, 3, 2, 6, 4$, and 5; and then 1 appears again, beginning the cycle anew. Since the **period length,** the number of digits in the period, is 6, moving the decimal point in the dividend and in the quotient six places to the right shows clearly that 7 divides $10^6 - 1$.

The computations shown in this example can be carried out for an arbitrary prime — that is, in the partial division of 1 by a prime p other than 2 or 5, a succession of remainders will appear and because there are only $p-1$ possible remainders in the division process, it is clear that a repetition must occur. A few attempts or a little thought will indicate that the first repetition will be a 1, and if this first repetition occurs at the mth step, it follows that p divides $10^m - 1$. This gives another proof that every prime other than 2 or 5 is a divisor of some repunit. If the long division of 1 by p is carried out to more places, the original sequence of remainders representing the repetend will be repeated — with a remainder of 1 appearing after km steps, $k = 1, 2, 3, \ldots$. From this fact it is clear that if p divides $10^n - 1$, then n must be a multiple of m, where m is the period length of $1/p$. This proves that if p is a prime divisor of R_n, then p is a primitive prime divisor of R_m, where m divides n.

The process described above shows the following: if p is a prime other than 2, 3, or 5, p will be a primitive prime divisor of R_n if and only if $1/p$ has a period length of n. For if $1/p$ has a period length of n, then n is the smallest number such that p divides $10^n - 1$, and this means that p is a primitive prime divisor of R_n. Each of these steps can be reversed to show the converse.

A look at the computation process above reveals that $2/p$, $3/p$, and all other proper fractions whose denominator is prime p have period lengths which are the same as that of $1/p$. So we shall designate the period length of any fraction whose denominator is prime p by $\lambda(p)$. Since $1/3 = .\bar{3}$, we have $\lambda(3) = 1$. If p is a primitive prime divisor, other than 3, of R_n, then $\lambda(p) = n$. Henceforth, we shall refer to $\lambda(p)$ as the period length of the prime p.

In summary, we see that every number n is the period length of some prime, and the primes having this period length are precisely the primitive prime divisors of R_n. Thus, for example, 11 is the only prime with period length 2, 37 is the only prime of period length 3, 333667 is the only prime of period length 9, 7 and 13 are the only primes of period length 6, etc. Thus the problem of determining the R_n for which p is primitive is precisely the problem of finding the period length $\lambda(p)$.

Finding the period length $\lambda(p)$ for a prime p other than 2 or 5 can be simplified by the use of congruence arithmetic. Recall that a is congruent to b modulo c, written $a \equiv b \pmod{c}$, iff $a - b$ is divisible by c, so that both a and b are members of an infinite set of numbers, called a congruence class, obtained by substituting integers for n in the expression $cn + b$. In the long division of 1 by 7 above, the successive remainders were $3, 2, 6, 4, 5, 1, \ldots$. These can be nicely expressed in the congruence notation just introduced by the congruences

$$10 \equiv 3 \pmod{7},$$
$$10^2 \equiv 2 \pmod{7},$$
$$10^3 \equiv 6 \pmod{7},$$
$$10^4 \equiv 4 \pmod{7},$$
$$10^5 \equiv 5 \pmod{7},$$
$$10^6 \equiv 1 \pmod{7}, \text{ etc.}$$

Suppose we wish to find the repunit for which 19 is a primitive divisor (equivalently, suppose we wish to find $\lambda(19)$). We can make use of the following property of congruences: if $x \equiv y \pmod{c}$ and $u \equiv v \pmod{c}$ then $xu \equiv yv \pmod{c}$. We begin by noting that

$$10 \equiv 10 \pmod{19}, \text{ and}$$
$$10^2 \equiv 5 \pmod{19}.$$

It follows from the preceding property of congruences that $10^3 \equiv 50 \pmod{19}$, or

$$10^3 \equiv 12 \pmod{19}.$$

Continuing, we get

$$10^4 \equiv 120 \pmod{19}, \quad \text{or} \quad 10^4 \equiv 6 \pmod{19},$$
$$10^5 \equiv 60 \pmod{19}, \quad \text{or} \quad 10^5 \equiv 3 \pmod{19},$$
$$10^6 \equiv 30 \pmod{19}, \quad \text{or} \quad 10^6 \equiv 11 \pmod{19},$$
$$10^7 \equiv 110 \pmod{19}, \quad \text{or} \quad 10^7 \equiv 15 \pmod{19},$$
$$10^8 \equiv 150 \pmod{19}, \quad \text{or} \quad 10^8 \equiv 17 \pmod{19},$$
$$10^9 \equiv 170 \pmod{19}, \quad \text{or} \quad 10^9 \equiv 18 \pmod{19}.$$

Since $18 \equiv -1 \pmod{19}$, it follows that $10^{18} \equiv (-1)(-1) \equiv 1 \pmod{19}$. Thus 19 is a prime divisor of R_{18}. Furthermore, it must be a primitive prime divisor of R_{18}, since we see above that it is not a prime divisor of R_m for any m dividing 18.

In some cases the period length of a prime can be determined even more easily because the even-odd parity of period lengths is a function of the congruence classes to which the primes belong. In order to explain this most economically, we make use of the following definition: the **residue index** $i(p)$ of a prime p is the quotient $(p-1)/\lambda(p)$. Many sources ([1], for example) show that, of the 16 congruence classes (mod 40) which contain primes, the primes in 8 of them are quadratic residues of 10 and the primes in the other 8 are not. If prime p is a quadratic residue of 10, then p divides $10^{(p-1)/2}-1$, and $\lambda(p)$ divides $(p-1)/2$ so $(p-1)/2\lambda(p)$ is an integer; call it k. Then $i(p)=(p-1)/\lambda(p)=2k$, so that $i(p)$ is an even number. If prime p is a quadratic non-residue of 10, then p divides $10^{(p-1)/2}+1$ but cannot divide $10^{(p-1)/2}-1$, because no prime divides two consecutive odd numbers. Prime p does, of course, divide the product of these two binomials, which is $10^{(p-1)}-1$. Consequently, $\lambda(p)$ divides $(p-1)$, but $\lambda(p)$ does not divide $(p-1)/2$. Since $i(p)$ is an integer, but half of it, $(p-1)/2\lambda(p)$, is not an integer, $i(p)$ must be an odd number. The set of primes is equally distributed among the 16 congruence classes (mod 40); so half of all primes have even residue indexes and the other half have odd residue indexes. In fact, if a prime is congruent to $\pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$, its residue index is odd; and if it is congruent to $\pm 1, \pm 3, \pm 9$, or $\pm 13 \pmod{40}$, its residue index is even. All other congruence classes (mod 40) are divisible by 2 and /or 5, and therefore they contain no primes which produce repeating decimals.

Because p is an odd prime, $p-1$ is even. There are three cases to consider:

1. $i(p)$ is odd;
2. $i(p)$ is even and $p-1 \equiv 2 \pmod 4$; and
3. $i(p)$ is even and $p-1 \equiv 0 \pmod 4$.

In the first case $\lambda(p)=(p-1)/i(p)$ is even and this happens when p is congruent to 7, 11, 17, 19, 21, 23, 29, or 33 (mod 40). In the second case, $i(p)$ must be congruent to 2 (mod 4) for $\lambda(p)$ to be an integer, and $\lambda(p)$ necessarily turns out to be an odd number. That is the case when p is congruent to 3, 27, 31, or 39 (mod 40). In the third case, $\lambda(p)$ is odd or even, depending upon whether or not the largest power of 2 that divides the residue index is the same as that which divides $p-1$.

With this knowledge at hand, some period lengths can be determined immediately. For example, suppose that a prime p is 1 more than twice another odd prime q, and consider the identity $i(p)=(p-1)/\lambda(p)$. Suppose that p has even period length. Then $\lambda(p)$ is either 2 or $p-1$ since these are the only even divisors of $p-1$. If p is different from 11 (the only prime whose period length is 2), then $\lambda(p)$ must equal $p-1$. An example is 383, a prime which is 1 more than twice the prime 191, and which is congruent to 23 (mod 40), meaning that it must have even period length. Immediately we can say that $\lambda(383)=382$. Suppose that p has odd period length. Then $\lambda(p)$ is either 1 or q since these are the only odd divisors of $p-1$. If p is different from 3 (the only prime whose period length is 1), then $\lambda(p)$ must be q. An example is 359, a prime which is 1 more than twice prime 179. It is congruent to 39 (mod 40), so it must have odd period length. Then $\lambda(359)=179$.

A prime p whose period length is $p-1$, like 7 and 383 already noted, is called a **full period prime.** That is because its period is as full, or as long, as it can be, since there are only $p-1$ possible integer remainders in the ordinary division-type determination of the period length of a prime. If

$\lambda(p) = p - 1$, then $i(p) = 1$, which is odd, so that all full period primes have odd residue indexes and even period lengths.

It follows from a theorem in [7] that all primes are equally distributed among the sixteen classes (mod 40) which contain primes. It has also been proved that twice as many primes are primitive divisors of repunits with even subscripts, as are primitive divisors of repunits with odd subscripts [8]. Counts taken over large sets of consecutive primes agree very closely with this generalization. It has not yet been proved that the set of full period primes is infinite, but I anticipate that this long-standing conjecture will become a theorem in the near future. But it has been shown that if there is an infinity of full period primes, then almost three-eighths of all primes are full period primes. The conjectured figure is actually about one-third of one percent less than three-eighths, and actual counts among large sets of consecutive primes agree very closely with this approximation.

Using these very good approximations, we obtain interesting ratios. Every full period prime has an even period length. So 3/8 or 9/24 of all primes are full period primes, and 1/3 or 8/24 of all primes have odd period lengths, leaving 7/24 of all primes as those which are not full period primes, but have even period lengths. Then the ratios of the numbers in these categories, to each other in the same order, are 9:8:7.

Divisibility of Repunits by Powers

Under what conditions is a repunit in base 10 divisible by a square? No one has yet found any repunits in base 10 with odd prime subscripts which are divisible by squares, but nobody has yet proved that such repunits are square-free. In base 3, $R_5 = 11111$ is divisible by 11^2; that is, $(3^5 - 1)/(3 - 1)$, which is 121, is clearly not square-free. That makes one reluctant to conjecture about the possibilities in other bases, even though no such counterexample is known in base 10.

Although 3 and its square are both primitive divisors of $10^1 - 1$, there are only two known primes which are primitive divisors of the same repunits in base 10 as their squares. They are 487 and 56598313, and both are full period primes. The latter was discovered during the past decade. Aside from such exceptions, the square of any prime p will divide a repunit only if its subscript is a multiple of $p\lambda(p)$. For example, the smallest repunit that 11^2 divides is R_{22}, because $\lambda(11) = 2$; and the other repunits that are divisible by 11^2 are R_{44}, R_{66}, R_{88}, etc. Since no square of an integer ends in 11, no repunit can be a perfect square.

What about divisibility of repunits in base 10 by higher powers? If $np^{m-1}\lambda(p)$ is the subscript of a repunit where m and n are positive integers and p is prime less than 2^{28} (which is as high as current research has gone), then the repunit is divisible by p^m, except for 3, 487, and 56598313. For example, 11^3 divides R_{242}, R_{484}, etc., and the smallest repunit divisible by 11^4 is R_{2662}. If p is 487 or 56598313, then p^m divides repunits with subscripts equal to $np^{m-2}\lambda(p)$, where $m \geqq 2$.

Primitive Cofactors

From both a practical and a recreational point of view, it is intriguing to realize that primitive cofactors have *patterns* which can be described and detailed without having to go through computations involving large numbers. Here are some samples. Except when n is the product of a prime and its period length, the following rules (all found in [9]) apply when n is composite. If $n = 2pc$, where p is an odd prime and c is not divisible by any primes other than 2 or p, then the primitive cofactor P_n of R_n is the concatenation of $(p - 1)/2$ strings of c 9's followed by c 0's, with the last 0 replaced by a 1. For example, when n is 44, $n = 2 \times 11 \times 2$, so that $(p - 1)/2 = 5$, and primitive cofactor P_{44} is 9900 9900 9900 9900 9901. When n is 50, then $n = 2 \times 5 \times 5$, so $(p - 1)/2 = 2$, and primitive cofactor P_{50} is 99999 00000 99999 00001.

If $n = 3^b$, where b is 2 or more, then the primitive cofactor P_n of R_n is the concatenation of 3^{b-1} 3's followed by 3^{b-1} 6's, with the last 6 replaced by a 7. For instance, $P_9 = 333667$, which, incidentally, is prime, and is therefore the only prime with a period length of 9. Also, since $81 = 3^4$,

$$P_{81} = 333333333\ 333333333\ 333333333\ 666666666\ 666666666\ 666666667.$$

Here is one more cute pattern. If $n = b^2$, and b is a prime other than 3, then the primitive cofactor P_n is the concatenation of $b-1$ strings of a 1 followed by $b-1$ 0's, all of which is followed by a 1. For example, $P_4 = 101$, which happens to be prime and is therefore the only prime with a period length of 4, as noted earlier. Also, since $49 = 7^2$,

$$P_{49} = 1000000\ 1000000\ 1000000\ 1000000\ 1000000\ 1000000\ 1.$$

Repunit Riddles

We close with some captivating puzzle-type problems which have appeared among sets of so-called Repunit Riddles in the *Journal of Recreational Mathematics.*

1. What digit does each letter of this multiplication represent?

$$\begin{array}{r} RRRRRRR \\ \times\ \ RRRRRRR \\ \hline \text{REPUNITINUPER} \end{array}$$

2. A car dealer's total receipts for the sale of new cars last year came to \$1, 111, 111.00. If each car had more than four cylinders and was sold for the same price as every other car, how many cars did he sell?
3. What are the smallest and largest primes which could have period lengths equal to k?

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Repunit Riddle Resolutions

1. The fact that the square of a repdigit is a palindrome suggests that the repdigit could be a repunit:

$$\begin{array}{r} 1111111 \\ \times\ \ 1111111 \\ \hline 1234567654321 \end{array}$$

2. This is a variation of a problem posed by Beiler and others. \$1,111,111.00 is R_7 expressed in dollars. There are only two prime divisors of R_7. They are 239 and 4649. New cars with more than four cylinders cost more than \$239, so the dealer must have sold 239 cars at \$4649.00 each.
3. The largest period length that a prime p can have is $p-1$. So if a prime has a period length of k, the smallest that prime could be is $k+1$. Every prime with period length k must be a divisor of repunit R_k and of no smaller repunit. The largest such divisor would be R_k itself. If no proper divisor of R_k has period length k, then R_k must be prime, and it is the largest prime that could have a period length of k.

Tiling the Plane with Congruent Pentagons

A problem for anyone to contribute to: a survey of the growing but incomplete story of pentagonal tilings of the plane.

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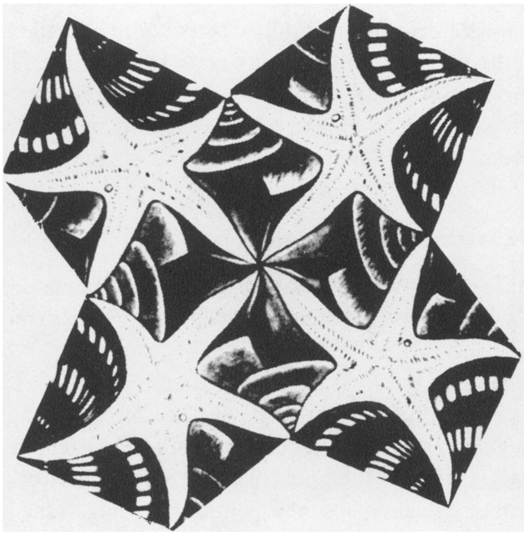
The importance of recreational mathematics and the involvement of amateur mathematicians has been dramatically demonstrated recently in connection with the problem of tiling the plane with congruent pentagons. The problem is to describe completely all pentagons whose congruent images will tile the plane (without overlaps or gaps). The problem was thought to have been solved by R. B. Kershner, who announced his results in 1968 [**18**], [**19**]. In July, 1975, Kershner's article was the main topic of Martin Gardner's column, "Mathematical Games" in *Scientific American.* Inspired by the challenge of the problem, at least two readers attempted their own tilings of pentagons and each discovered pentagons missing from Kershner's list. New interest in the problem has been aroused and both amateur and professional mathematicians are presently working on its solution.

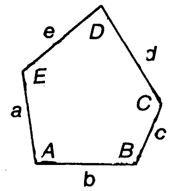


Tilings by convex polygons

A polygon is said **to tile** the plane if its congruent images cover the plane without gaps or overlaps. The pattern formed in this manner is called a **tiling** of the plane, and the congruent polygons are called its tiles. A **vertex of the tiling** is a point at which 3 or more tiles meet. It is well known that any triangle or any quadrilateral can tile the plane. In fact, any single triangle or quadrilateral can be used as a "generating" tile in a tiling of the plane which is **tile-transitive** (isohedral). This simply means that the generating tile can be mapped onto any other tile by an isometry of the tiling. The translations, rotations, reflections, and glide-reflections which map a tiling onto itself make up the **symmetry group** of the tiling. Thus, in terms of group theory, a tile-transitive tiling is one whose symmetry group acts transitively on the tiles.

Also, for an arbitrary triangle or quadrilateral there always exists a tiling which in addition is **edge-to-edge.** This means that for any two tiles, exactly one of the following holds: (i) they have no points in common, (ii) they have exactly one point in common, which is a vertex of each tile (such a point is also a vertex of the tiling), (iii) their intersection is an edge of each tile. Typical tilings of triangles and of quadrilaterals having these properties are shown in [**5**], [**25**].

1. In a given tiling, pentagons marked with a dot are oppositely congruent to those which are unmarked, i.e., the plain tiles are 'face up', and the marked tiles are 'face down'.
2. In each tiling there is outlined a minimal block of one or more pentagons which generates the tiling when acted on by the symmetry group of the tiling.
3. Angles A, B, C, D, E of one pentagonal tile are identified in each tiling for use with Tables I, II, III. Sides a, b, c, d, e of that tile correspond to the following labeling:

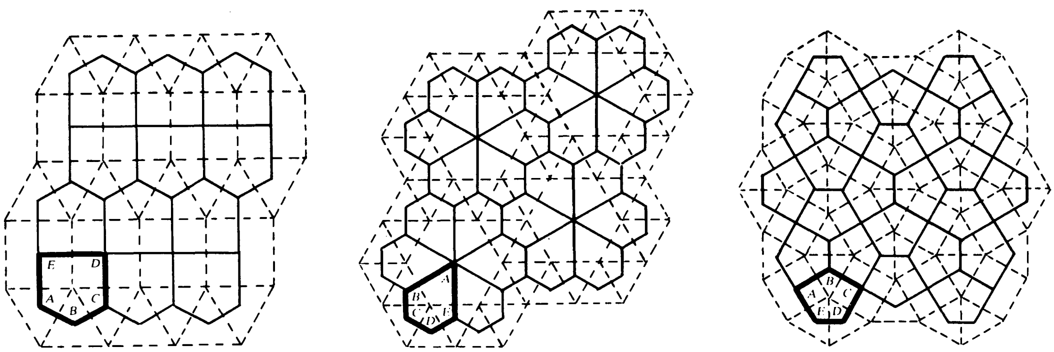


Key to all diagrams

If we ask the natural question, "Do convex polygons of five or more sides tile the plane?", it is clear that the obvious general answer is "Not always." It is clear that not every convex pentagon tiles the plane — a regular pentagon is a prime example. However, regular hexagons do tile the plane (how many times have you seen this pattern on a 1930's bathroom floor?), but not all hexagons tile. A complete description of all hexagons which do tile the plane was discovered independently by several mathematicians and is discussed in **[2]**, **[5]**, **[14]**, **[18]**, **[19]**, **[28]**. It can also be demonstrated that no convex polygon of more than six sides can tile the plane. Thus, the problem of describing all convex pentagons which tile the plane is the only unanswered part of our question. In what follows, we make several observations related to the pentagonal tiling problem and report on the most recent contributions to its solution.

Discovering tilings by pentagons

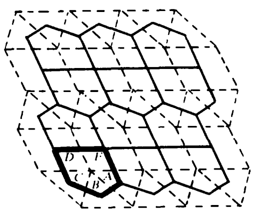
Three of the oldest known pentagonal tilings are shown in FIGURE 1. As Martin Gardner observed in **[5]**, they possess "unusual symmetry". This symmetry is no accident, for these three tilings are the duals of the only three Archimedean tilings whose vertices are of valence 5. The underlying Archimedean tilings are shown in dotted outline. Tiling (3) of FIGURE 1 has special aesthetic appeal. It is said to appear as street paving in Cairo; it is the cover illustration for Coxeter's *Regular Complex Polytopes*, and was a favorite pattern of the Dutch artist, M. C. Escher. Escher's sketchbooks reveal that this tiling is the unobtrusive geometric network which underlies his beautiful "shells and starfish" pattern. He also chose this pentagonal tiling as the bold network of a periodic design which appears as a fragment in his 700 cm. long print "Metamorphosis II."



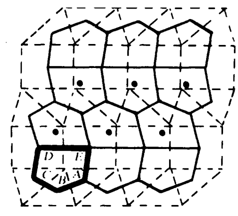
(1) (2) (3)

The three pentagonal tilings which are duals of Archimedean tilings. The underlying Archimedean tiling is shown in dotted outline.

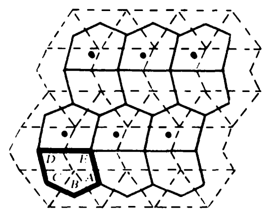
FIGURE 1.



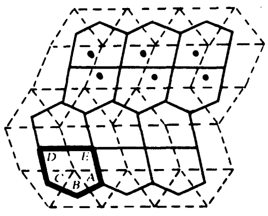
(4)



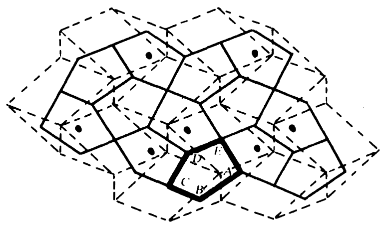
(5)



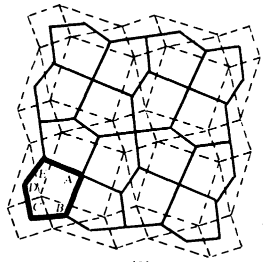
(6)



(7)



(8)



(9)

Tile-transitive, edge-to-edge tilings by pentagons, obtained as duals of vertex-transitive tilings (shown in dotted outline). Only in the case of (6) is a non-convex tile necessary in the underlying tiling. Tiling (8) is discussed in [4].

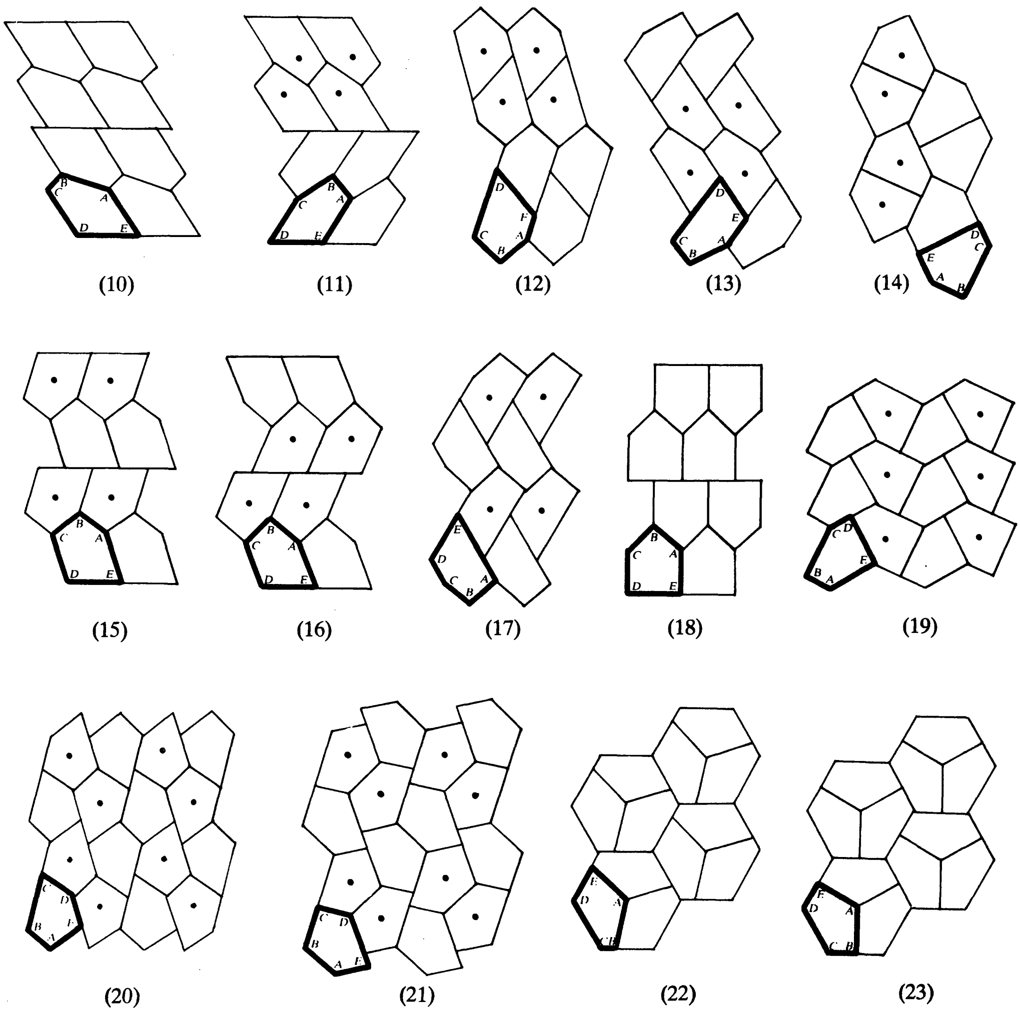
FIGURE 2.

Tiling (3) can also be obtained in several other ways. Perhaps most obviously it is a grid of pentagons which is formed when two hexagonal tilings are superimposed at right angles to each other. F. Haag noted that this tiling can also be obtained by joining points of tangency in a circle packing of the plane [**12**]. It can also be obtained by dissecting a square into four congruent quadrilaterals and then joining the dissected squares together [**26**]. The importance of these observations is that by generalizing these techniques, other pentagonal tilings can be discovered.

The three Archimedean tilings which have as duals the pentagonal tilings in FIGURE 1 are vertex transitive tilings. An edge-to-edge tiling by polygons is called **vertex transitive** (isogonal) if the symmetry group of the tiling is transitive on the vertices of the tiling. FIGURE 2 shows six other edge-to-edge pentagonal tilings that arise as duals of vertex-transitive tilings of the plane. Recently, B. Grünbaum and G. C. Shephard showed that the nine tilings of FIGURES 1 and 2 are the only distinct "types" of pentagonal tilings which are edge-to-edge and tile-transitive. Roughly speaking, two "types" of tilings will differ if they have different symmetry groups or if the relationship of tiles to their adjacent tiles differs. Details are given in [**6**]. In addition, these mathematicians have classified all vertex-transitive tilings, and their list shows that no such tiling by convex pentagons is possible [**7**].

If we begin with a tiling by congruent convex hexagons, then pentagonal tilings can arise in two different ways. First, it may be possible to superimpose the hexagonal tiling on itself so as to produce a tiling by congruent pentagons. Tilings (8) and (9) of FIGURE 2 can arise in this way. Also, beginning with a hexagonal tiling, it may be possible to dissect each hexagon into two or more congruent pentagons, thus producing a pentagonal tiling. Many tilings of FIGURES 1, 2, and 3 can be viewed in this manner. Three other examples of such tilings given in FIGURE 4 also serve to illustrate other properties of pentagonal tilings that can occur. Note that (24) is tile-transitive but not edge-to-edge, tiling (25) is edge-to-edge but not tile-transitive, and tiling (26) is neither edge-to-edge nor tile-transitive.

Finally, experimentation in fitting pieces together or adding or removing lines from other geometric tilings can lead to the discovery of pentagonal tilings. FIGURE 5 illustrates this with two tilings by a simple "house" shape pentagon.



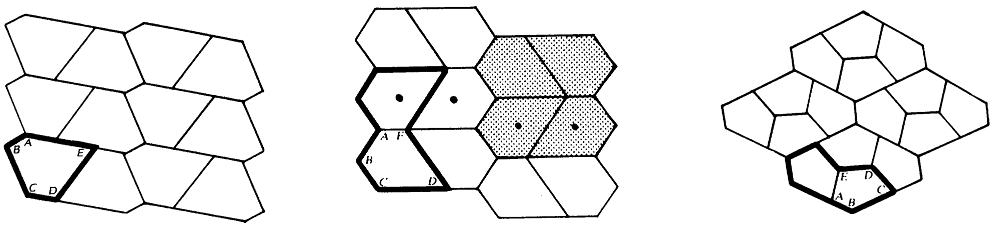
Tile-transitive tilings by pentagons which are not edge-to-edge.

FIGURE 3.

Methodical attacks

Pentagonal tilings appear as illustrations in several early papers which explore the general problem of classifying plane repeating patterns (especially [**11**]), but it appears that the first methodical attack on classifying pentagons which tile the plane was done in 1918 by K. Reinhardt in his doctoral dissertation at the University of Frankfurt [**28**]. He discovered five distinct types of pentagons, each of which tile the plane. More precisely, he stated five different sets of conditions on angles and sides of a pentagon such that each set of conditions is sufficient to ensure that (i) a pentagon fulfilling these conditions exists, and (ii) at least one tiling of the plane by such a pentagon exists. Each of these five sets of conditions defines a *type* of pentagon; pentagons are considered to be of different types only if they do not satisfy the same set of conditions. Many distinct tilings can exist for pentagons of a given type. Reinhardt no doubt hoped that his five types constituted a complete solution to the problem, but he was unable to show that a tiling pentagon was necessarily one of these types.

Each of the five types described by Reinhardt (called types 1 — 5 in TABLE I) can generate a tile-transitive tiling of the plane. His thesis completely settled the problem of describing all convex



(24) (25) (26)

Tilings by pentagons obtained by dissecting hexagonal tilings. Tiling (24) is tile-transitive, but in (25) and (26), it is impossible to map one pentagon in an outlined block onto the other pentagon in that block by a symmetry of the tiling. The shaded portion of (25) shows a "double hexagon" which has been dissected into 4 congruent pentagons.

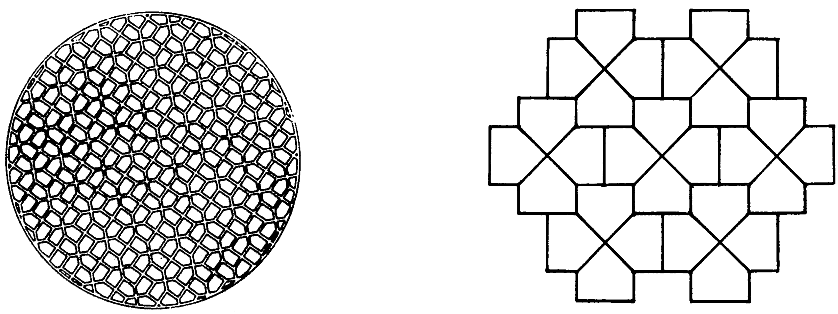
FIGURE 4.

hexagons which tile the plane — there are just three types, and each of these can generate a tile-transitive tiling. The fact that if a hexagon can tile the plane at all, then that same hexagon can generate a tile-transitive tiling, considerably simplifies the hexagonal tiling problem. Unfortunately, this result is not true for pentagons and this may be the reason that Reinhardt did not pursue the problem further by trying to find other types of pentagons which tile.

In [**14**], pp. 81–91, Heesch and Kienzle methodically explore the problem of describing types of pentagons which can generate tile-transitive tilings and affirm that Reinhardt's five types are the only convex ones possible. Most recently, B. Grünbaum and G. C. Shephard, using a classification scheme for tilings, found exactly 81 "types" of tile-transitive tilings of the plane by quite general "tiles" [**6**], and, as a result, have not only confirmed these earlier results, but also have shown that there are exactly twenty-four distinct "types" of tile-transitive tilings by pentagons according to their classification scheme [**9**]. Nine of these are edge-to-edge (tilings (1)–(9)); the other fifteen are illustrated by tilings (10)–(24). TABLE I summarizes information on these tilings.

In 1968, R. B. Kershner of Johns Hopkins University announced that there are 8 types of pentagons which tile the plane [**18**], [**19**]. He devised a method different from Reinhardt's for classifying pentagons which can tile, and this scheme left out any assumptions of tile transitivity for associated tilings. Happily, his search yielded three classes of pentagons not on Reinhardt's list. His 3 new pentagonal tiles (types 6, 7, 8 on TABLE II) each have an associated tiling which is edge-to-edge and not tile-transitive (FIGURE 6).

Although Kershner's claim — that these three additional types of pentagons completed the list of



(27) (28)

A "house shape" pentagon tiles in a variety of ways. Tiling (27) is a Chinese lattice design (*Chinese Lattice Designs*, D. S. Dye, Dover, 1974, p. 340); (28) is a familiar geometric pattern of interlocked St. Andrews crosses. Other patterns are (1), (14), (18).

FIGURE 5.

Type, with characterizing conditions	Tile-Transitive tilings	Additional Conditions necessary for tiling	International Notation for Symmetry Group of Tiling (see [30])	Isohedral type (from [6])	Type from [9]	Other tilings shown
1. $D+E=\pi$	(24)		$p2$	$IH4$	P_5-4	
	(10)	$a=d$	$p2$	$IH4$	P_5-5	
	(11)	$a=d$	pgg	$IH5$	P_5-8	
	(4)	$a=d$	$p2$	$IH23$	P_5-18	
	(5)	$a=d$	pmg	$IH24$	P_5-19	
	(12)	$b=c$	pgg	$IH5$	P_5-6	(25); (33) if $a=e, b=d$, $D=E=\pi/2$
	(13)	$a+e=d$	pgg	$IH5$	P_5-7	
	(14)	$a+d=c$	pgg	$IH6$	P_5-12	
	(15)	$a=d, b=c$	pg	$IH2$	P_5-1	(35) if $A+2D=2\pi$, $a=b=c=d$
	(16)	$a=d, b=c$	pgg	$IH5$	P_5-9	
	(6)	$a=d, b=c$	cm	$IH22$	P_5-17	(40), (41) if $D=80°$ and $a=b=c=d=e$
	(7)	$a=d, b=c$	pgg	$IH25$	P_5-20	
	(17)	$b=c, a=d+e$	pg	$IH2$	P_5-2	
	(18)	$D=E=\pi/2, A=C,$ $a=d, b=c$	pmg	$IH15$	P_5-14	
	(1)	$D=E=\pi/2, A=C,$ $a=d, b=c$	cmm	$IH26$	P_5-21	(27), (28) if $B=\pi/2$
	(19)	$D+B=\pi, c=e,$ $a=b+d$	pg	$IH3$	P_5-3	
2. $C+E=\pi$	(20)		pgg	$IH6$	P_5-10	(26) if $A+C=\pi$, $d=e$
$a=d$	(21)		pgg	$IH6$	P_5-11	
	(8)	$c=e$	pgg	$IH27$	P_5-22	(36) if $D+2E=2\pi$ $a=c=d=e$
3. $A=C=D=2\pi/3$	(22)		$p3$	$IH7$	P_5-13	
$a=b$	(23)	$B=E=\pi/2$	$p31m$	$IH16$	P_5-15	
$d=c+e$						
4. $A=C=\pi/2$	(9)		$p4$	$IH28$	P_5-23	(27) if $E=\pi/2$ and $a=b=e$
$a=b$ $c=d$	(3)	$D=E$	$p4g$	$IH29$	P_5-24	
5. $A=\pi/3$	(2)		$p6$	$IH21$	P_5-16	
$C=2\pi/3$						
$a=b, c=d$						

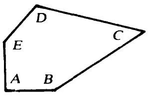
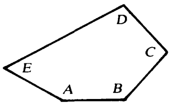
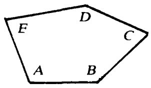
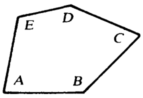
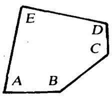
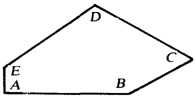
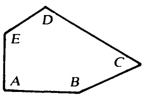
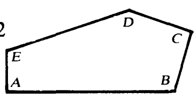
THE FIVE TYPES OF PENTAGONS which can generate tile-transitive tilings of the plane. The tiles for a given transitive tiling are not always uniquely of one type (e.g. the tiles of tiling (19) satisfy conditions on angles and sides of both types 1 and 2). In this table we have listed each transitive tiling only once, thus identifying its tiles by just one type.

TABLE I

pentagons which tile — was later shown false, still his discovery was important. It confirmed that there are pentagons whose associated tilings cannot be tile-transitive, thereby answering a question raised by J. Milnor in [**24**, p. 499]. His search for pentagons which tile had been a methodical one; yet still, in his own words, he "made at least 2 errors, one of commission, and one of omission".

Contributions by amateurs

The publication of the problem and Kershner's list by Martin Gardner in [**5**] stimulated amateurs to try to find pentagons which tile. Richard James III, a California computer scientist, read the

Type, with characterizing conditions	Illustrative tile and tiling number	Symmetry group of tiling		Remarks
6. $C+E=\pi$ $A=2C$ $a=b=e$ $c=d$		(29)	$p2$	2-block transitive. Associated block tiling (29)–B is isohedral type $IH4$.
7. $2B+C=2\pi$ $2D+A=2\pi$ $a=b=c=d$		(30)	pgg	2-block transitive. Associated block tiling (30)–B is isohedral type $IH6$.
8. $2A+B=2\pi$ $2D+C=2\pi$ $a=b=c=d$		(31)	pgg	2-block transitive. Associated block tilings (31)–B and (31)–B' are type $IH6$.
9. $2E+B=2\pi$ $2D+C=2\pi$ $a=b=c=d$		(34)	pgg	2-block transitive. Associated block tiling (34)–B is type $IH53$.
10. $E=\pi/2$ $A=\pi-D$ $B=\frac{\pi+D}{2}$ $C=\frac{\pi-D}{2}$ $a=e=b+d$		(32) (33) if $D=\pi/2$, $b=d$	$p2$ cmm	D is bounded: $\pi-\tan^{-1}(4/3)<D<\tan^{-1}(4/3)$ 3-block transitive. Associated block tilings (32)–B and (33)–B are type $IH4$.
11. $A=\frac{\pi}{2}$ $C+E=\pi$ $2B+C=2\pi$ $d=e=2a+c$		(37)	pgg	2-block transitive. Associated block tiling is type $IH6$.
12. $A=\frac{\pi}{2}$ $C+E=\pi$ $2B+C=2\pi$ $e+c=d=2a$		(38)	pgg	2-block transitive. Associated block tiling is type $IH6$.
13. $A=C=\pi/2$ $B=E=\pi-D/2$ $c=d$ $2c=e$		(39)	pgg	2-block transitive. Associated block tiling is IH5.

Eight Types of Pentagons which Tile, but for Which No Tile-Transitive Tiling Exists.

TABLE II.

problem and decided not to look at Kershner's list, but see if he could find some pentagonal tilings himself. Familiar with the common tiling by octagons and squares, and noting that an octagon is easily dissected into four congruent pentagons by perpendicular lines through its center, he attempted to change the familiar tiling into a pentagonal one. He was successful and sent an example to Martin Gardner [**17**]. FIGURE 7 shows two tilings by pentagons of James's type (type 10 on TABLE II). James's discovery served to point out a hidden assumption in Kershner's search — he had, in fact, only been looking for pentagons which could tile in an edge-to-edge manner, or in a manner in which every tile was surrounded by six vertices of the tiling (as, for example, in tilings (10) through (24)). James's pentagons are only capable of tiling in a manner which is not tile-transitive and not edge-to-edge. In addition, some pentagons in this tiling are surrounded by 5 vertices of the tiling, others are surrounded by 7 vertices of the tiling.

Marjorie Rice, a Californian with no mathematical training beyond "the bare minimum they required ... in high school over 35 years ago", also read Gardner's column and began her own methodical attack on the problem. Her approach was to consider the different ways in which the vertices of a single pentagon could "come together" to form a vertex of a tiling by congruent images of that pentagon. These considerations forced conditions on the angles and sides of the pentagon if it was to tile, thus giving either a description of a pentagon which could tile in a prescribed manner, or forcing the conclusion that no pentagon could be constructed which satisfied the conditions.

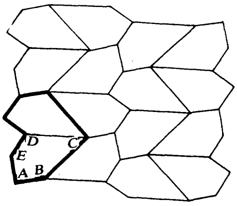
This essentially combinatorial search yielded over forty different tilings by pentagons and included a tiling by a new type of tile not on Kershner's list. Her discovery (type 9 on TABLE II) showed that Kershner's search (which was similar to hers) erroneously eliminated the possiblity of this type of edge-to-edge tiling. A later methodical search by Rice considered twelve different classes of pentagons, each class corresponding to a description of which sides of a given pentagon are equal. Possible tilings for each class were sought — and for every class at least one tiling was found. Over 58 diagrams of distinct tilings were produced in this effort, most of them non-transitive tilings by tiles of type 1. Even though she missed several of the 24 tile-transitive tilings, her scheme was complete enough to produce a tiling for every one of the pentagons of types 1–10 in TABLES I and II. No other new types of tiles were produced in this effort. FIGURE 8 shows three of Rice's tilings for the class of pentagons having four equal sides, including tiling (34) associated to her type 9. (The pentagons in tilings (30) and (31), FIGURE 6, also have 4 equal sides.)

Block transitive tilings

The solution of the hexagonal tiling problem was simplified by a theorem which reduced the problem to one of hexagons capable of producing tile-transitive tilings. Although this theorem is clearly false for the case of pentagons which tile, we can observe that for each pentagon of types 1 through 10, there exists a tiling containing a minimal 'block' of congruent pentagons which has the property that (i) the tiling consists of congruent images of this block and (ii) this block can be mapped onto any other congruent block by an isometry of the tiling. If a minimal such block contains n pentagons, we will say that the tiling is ***n*-block transitive.** Thus a tile-transitive tiling is 1-block transitive. We remark that for $n \geqq 2$, a given n-block transitive tiling may have several non-congruent minimal n-blocks. We have outlined two such 2-blocks in tiling (31).

If we remove the interior edges of pentagons in the heavily outlined blocks shown in the tilings in FIGURES 6, 7, and 8, the resulting transitive block tilings reveal information not immediately apparent from these pentagonal tilings alone. In FIGURE 9, we can see that each of Kershner's tilings (29), (30), (31) produces a block tiling in which each block is surrounded by six vertices of the tiling. Thus, these block tilings are formed by non-convex tiles which are topological hexagons. The Kershner tilings are obtained from these 'hexagonal' tilings by bisecting each 'hexagon' into two congruent pentagons. This method parallels the technique noted earlier of obtaining pentagonal tilings by bisecting hexagonal tilings. The reader can verify that the 2-block transitive tilings (25), (26), (35) are also obtained by bisecting blocks which tile as topological hexagons.

Tilings (32), (33) of James's pentagons also have associated block tilings of topological hexagons (FIGURE 9). In this case, however, each 'hexagon' has been dissected into 3 congruent pentagons, an occurrence that has no parallel in any of the pentagonal tilings known prior to James's discovery. It might be appropriate to note here that the symmetry group of the block tiling (33) – B of FIGURE 9 is $p2$, which is a proper subgroup of the symmetry group of the pentagonal tiling (33). (The names of tiling symmetry groups are given in TABLES I and II.) This occurrence is not surprising, since the removal of some edges of the pentagonal tiling can cause loss of some symmetries of the pattern (in this case, reflections).



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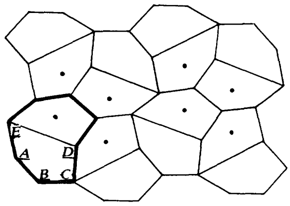

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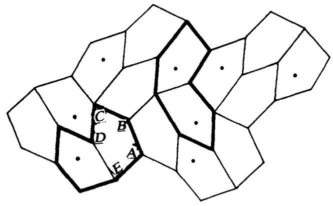
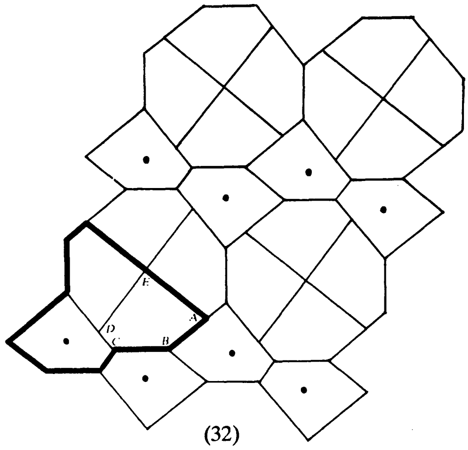


(31)

Tilings for each of the three types of pentagons discovered by R. B. Kershner. Each tiling is 2-block transitive, and there exists no tile transitive tiling for these types of pentagons. Two non-congruent 2-blocks are outlined in tiling (31); each of these generates a block-transitive tiling.

FIGURE 6.



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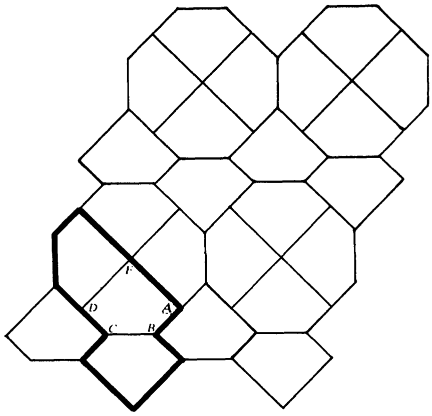
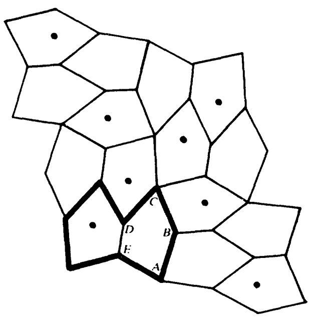


(33)

Tilings by type of pentagon discovered by Richard James III. The tilings consist of strips of attached octagons, separated by strips of bow ties. Each octagon contains four pentagons, and each bow tie contains two pentagons, which are in opposite orientation from those in the octagons. The tilings are 3-block transitive.

FIGURE 7.



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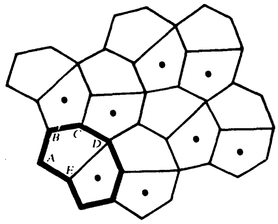
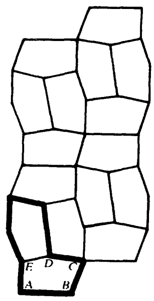


(35)



(36)

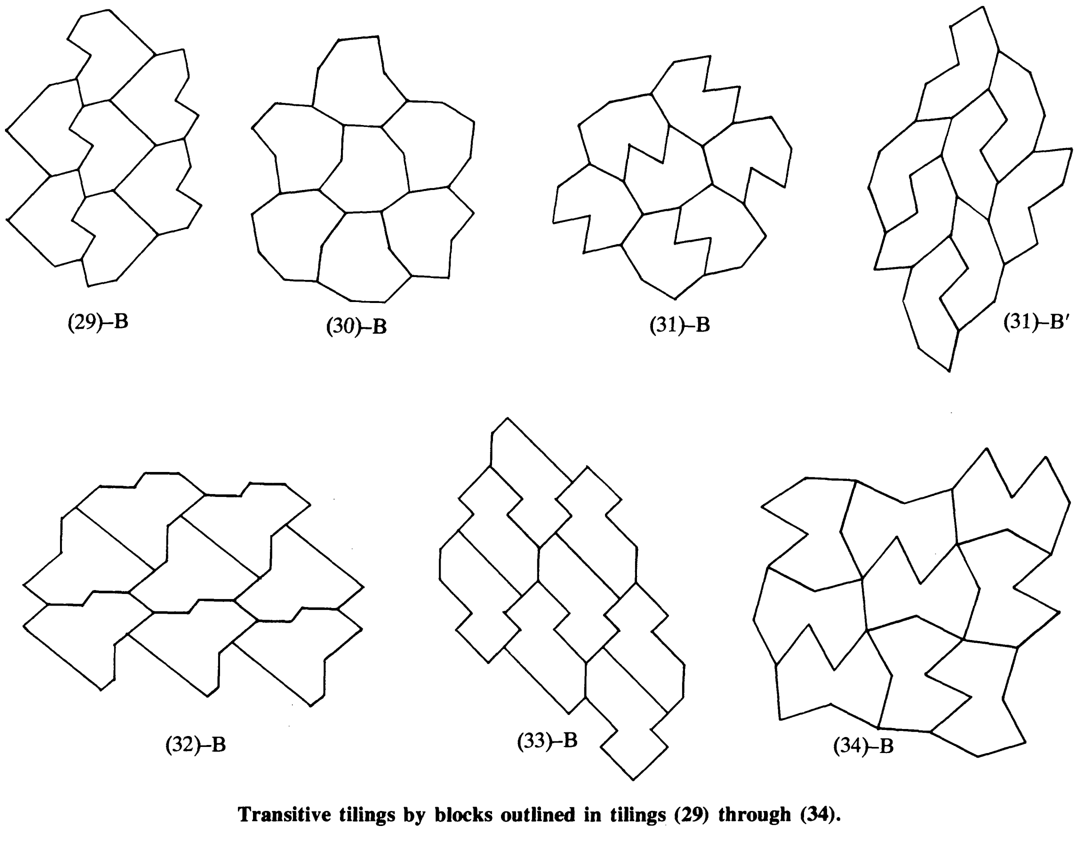
Three tilings discovered by Marjorie Rice, each containing congruent pentagons having four equal sides. The tile in (34) is type 9, and was a new addition to Kershner's list. The tile in (35) is type 1, the tile in (36) is type 2. All tilings are 2-block transitive, and for type 9, no tile-transitive tiling exists.

FIGURE 8.

FIGURE 9 shows the surprising fact that Rice's tiling (34) of type 9 pentagons has as its associated block tiling one in which each block is surrounded by just 4 vertices of the tiling. Thus tiling (34) is produced by dissecting a transitive tiling of topological quadrilaterals (Rice's tiling (36) is also obtained from 'quadrilateral' blocks). This dissection of a 'quadrilateral' tiling to produce a pentagonal tiling is most unexpected, since ordinary quadrilateral tiles (convex or non-convex), when bisected, produce either triangles or new quadrilaterals.

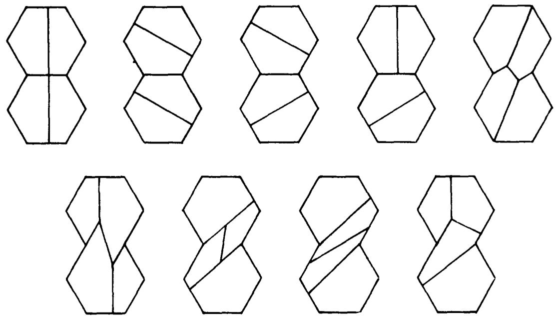
It is now easy to speculate that new types of pentagons which tile can be discovered by considering blocks of two or more congruent pentagons and determining if such blocks can tile transitively. A preliminary version of this article prompted Rice to examine a particular family of blocks, with hopes of determining new 2-block transitive pentagonal tilings. She observed that several of the 2-block transitive tilings previously discussed could also be viewed as tilings by blocks of four pentagons where these larger blocks had the outline of two hexagons stuck together. FIGURE 10 contains a schematic diagram of these "double hexagon" blocks, with the types of dissections of these blocks considered by Rice. Over sixty 2-block transitive tilings of pentagons were discovered in this way, some previously known ((25) and (26), for example), and some new. Best of all, two of the new tilings showed new types of pentagons! Just as this article was going to press, Marjorie Rice discovered yet another new tile as the result of a further search for new 2-block transitive tilings. FIGURE 10 shows the tilings associated to these new tiles (types 11, 12, and 13 on Table II).

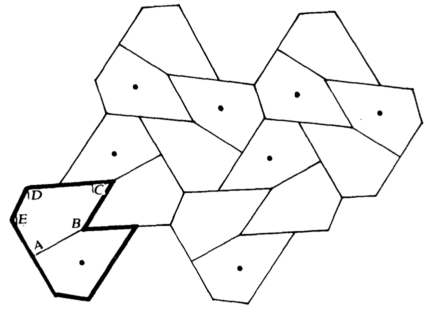
It is quite likely that still other new types of pentagons which tile can be discovered by considering dissections of transitive block tilings. The enumeration of the 81 types of isohedral tilings in [6] (complete with helpful diagrams) makes this task feasible. Checking possibilities will be an extremely



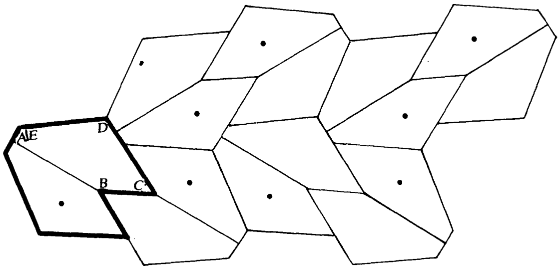
Transitive tilings by blocks outlined in tilings (29) through (34).

FIGURE 9.





(38)



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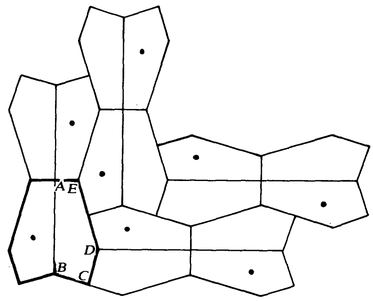


(39)

Marjorie Rice's schematic drawings of the "double hexagon" blocks of 4 pentagons. By adjusting sides and angles, many blocks of 4 congruent pentagons which tiled were found. Tilings (37) and (38) are for pentagons of types 11 and 12, respectively. Both are formed by blocks of the type shown at the lower left in the schematic diagram. Tiling (39) is for a type 13 pentagon, and is formed by blocks like those shown at the upper left in the schematic diagram.

FIGURE 10.

lengthy task, however, with a great deal of built-in repetition. This is assured by the fact that a given pentagonal tiling may have many distinct n-blocks which produce transitive block tilings and the additional fact that a single pentagonal tile may produce many distinct tilings. In order to determine if the list of types is complete, a theorem is needed to put some kind of bound on the possibilities. For the known pentagons which tile, there always exists an n-block transitive tiling for $n \leqq 3$. It is natural to hope that the following theorem is true: A pentagon tiles the plane only if there exists an n-block transitive tiling by that pentagon for $n \leqq 3$.

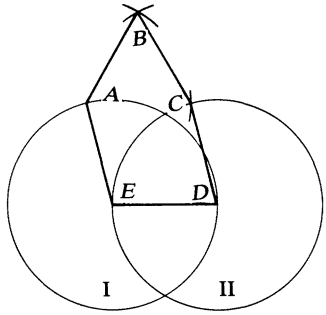
Equilateral pentagons which tile

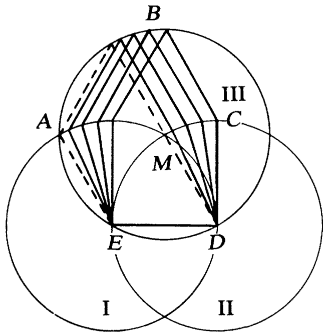
Although regular pentagons cannot tile the plane, a surprising variety of equilateral pentagons do. Of the 13 types of pentagons which tile it is obvious that types 3, 10, 11, 12, and 13 cannot be equilateral. Using construction techniques, together with familiar trigonometric relations, and the extension of these relations found in [**20**], we determined all possible equilateral pentagons of known types which tile. Types 1 and 2 provide distinct infinite families of equilateral pentagons. For types 4, 7 and 8, there is a unique equilateral pentagon of each type. Types 5, 6 and 9 cannot be equilateral. In order to investigate the equilateral case for types 7, 8 and 9, we studied the general class of equilateral pentagons $PQRST$ satisfying the condition $2P + Q = 2\pi$. FIGURE 11 contains information on this class. TABLE III contains a summary of details for all known types of equilateral pentagons which tile.

Since any two congruent equilateral pentagons will match edge to edge in any order, the possibilities for tilings are great. In attempting to determine all equilateral pentagons which tile, Marjorie Rice produced many interesting tilings. In addition, Martin Gardner's article inspired

TABLE III. Equilateral Pentagons Known to Tile. Types 1–2

Type 1. $D+E=\pi$





Construction

Choose angle E.
Construct sides $AE=ED$ by drawing circle I with center at E. Construct circle II with center D and radius DE. Find vertex C on circle II so that $DC \parallel AE$. Construct $CB=AB$. The pentagon $ABCDE$ has the outline of an equilateral triangle atop a rhombus. Draw circle III with radius equal to DE, and with center M, the intersection of circle I and circle II. Then for each pentagon $ABCDE$, vertex A lies on circle I, vertex C lies on circle II, and vertex B lies on circle III.

The diagram shows representatives of this type for $\frac{\pi}{2} \leqq E < \frac{2\pi}{3}$.

Angles

$\frac{\pi}{3} < E < \frac{2\pi}{3}$ $\qquad A = \frac{4\pi}{3} - E$ $\qquad C = \frac{\pi}{3} + E$

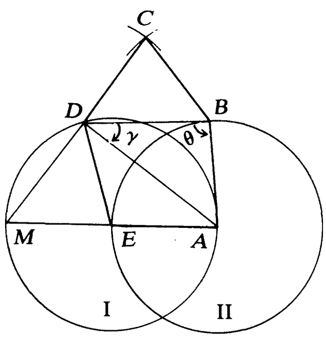
$B = \frac{\pi}{3}$ $\qquad D = \pi - E$

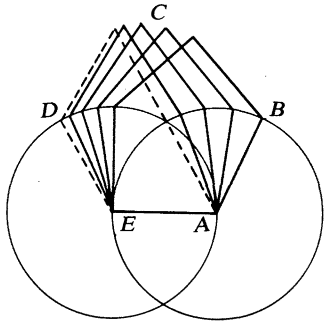
Illustrative tilings

Tile-transitive:
(4), (5), (6), (7), (10),
(11), (12), (15), (16), (24).
For $E = \frac{\pi}{2}$, (1) and (18).

Non-transitive:
(25)
For $D = 80°$, (35), (40), and (41).

Type 2. $C+E=\pi$





Construction

Choose angle E.
Construct sides $AE=ED$ by drawing circle I with center at E. Extend EA to M on circle I. Construct circle II with center A and radius AE. Find point B on circle II so that $DB=DM$; construct $\Delta DCB \simeq \Delta DEM$, (so $C=\pi-E$). Join AB to form pentagon $ABCDE$.

To obtain the description of the angles, draw segment DA, and denote $\gamma = \sphericalangle BDA$ and $\theta = \sphericalangle DBA$. Note ΔMDA is a right triangle, with $\sphericalangle DMA = E/2$ and $\sphericalangle DAM = C/2$. These facts, together with the law of cosines for ΔDBA lead to the angle relationships given.

The diagram shows representatives of this type for $\frac{\pi}{2} \leqq E < \frac{2\pi}{3}$.

Angles

$\frac{\pi}{3} < E < \frac{2\pi}{3}$

$C = \pi - E$

$\gamma = \arccos\left(\frac{3}{4\sin E}\right)$;

$\frac{\pi}{6} < \gamma \leqq \arccos(3/4)$

$\theta = \arccos\left(\frac{1+4\cos E}{4\cos E/2}\right)$

$D = \frac{\pi}{2} + \gamma$

$B = E/2 + \theta$

$A = \frac{3\pi}{2} - E/2 - \theta - \gamma$

Illustrative tilings

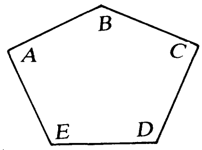
Tile-transitive:
(8), (20), (21).
Non-transitive: (36)

Special cases:
Types 4 and 8 below.

Types 4–7 **Equilateral Pentagons Known to Tile.** TABLE III.

Type 4.

$A = C = \frac{\pi}{2}$



Construction

Follow the construction of type 2 above for chosen angle $E = \pi/2$, then re-label the vertices of the pentagon replacing E by A, D by B, etc., in clockwise order. Angles can be established easily using right triangle relationships.

Angles

$A = C = \frac{\pi}{2}$ $\qquad B = 2\pi - 2D \sim 131°24'$

$D = E = \frac{\pi}{4} + \arccos\frac{\sqrt{2}}{4} \sim 114°18'$

Illustrative tiling: (3) and tilings listed for type 2 above.

Type 5.

$A = \frac{\pi}{3}, C = \frac{2\pi}{3}$

Impossible. The conditions $A = \pi/3$, $C = 2\pi/3$ imply that the pentagon would have to be of type 2 above. For $A = \pi/3$, the construction yields a limiting quadrilateral of the type 2 family, i.e., an equilateral "pentagon" with $E = \pi$.

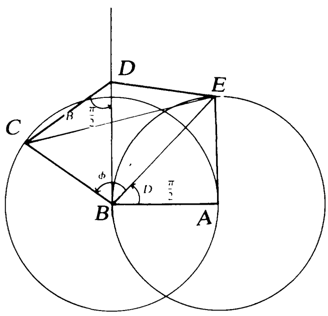
Type 6.

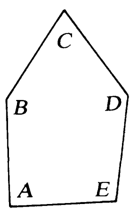
$C + E = \pi, A = 2C$

Impossible. The conditions $C + E = \pi$ and $A = 2C$ imply that the pentagon would have to be of type 2 above, with $\pi/3 < C < \pi/2$. Then $\pi/2 < E < 2\pi/3$ implies $-1 < \cos E < -1/2$ and $\cos E/2 > 0$, so $\arccos\,\theta < 0$. Thus $\theta > \pi/2$, and since $\gamma > \pi/6$, we have $3C/2 = A - C/2 = \pi - \theta - \gamma < \pi/3$, a contradiction to $\pi/3 < C$.

Type 7.

$2B + C = 2\pi,\ 2D + A = 2\pi$





Construction

Let $B = P$ and $C = Q$ in FIGURE 11; then $D = R$, $E = S$, $A = T$. Adjust vertex R on line l until $2D + A = 2\pi$ (a unique solution). Let $\phi = \sphericalangle CBE$. The angle relations indicated in figure 11, together with the additional condition $2D + A = 2\pi$, give the following general conditions on angles:

$B = \frac{\pi}{2} - \frac{A}{2} + \phi$ $\qquad D = \pi - A/2$

$C = \pi + A - 2\phi$ $\qquad E = \frac{\pi}{2} - A + \phi$

The extended law of cosines in [20] gives $\sin\phi = 1 - \cos A$; $\phi = \pi - \arcsin(\sin\phi)$. In addition, the following equations can be obtained, the first from FIGURE 11, the second by combining law of cosines for ΔCDE and law of sines for ΔCBE:

$$\cos C - \cos E = 1/2$$

$$16\sin^4 D + 8\sin^2 D + 4\cos D = 5$$

To obtain the angle approximations given, we noted by construction that A was close to 89°. We then computed the angles of the pentagon as A ranged over values within 1° of 89°, and computed the error in the above equations for this range. The angles given produce an error of less than .001 in both equations.

Angles

$A \sim 89°16'$ $\qquad C \sim 70°55'$ $\qquad E \sim 99°54'30''$

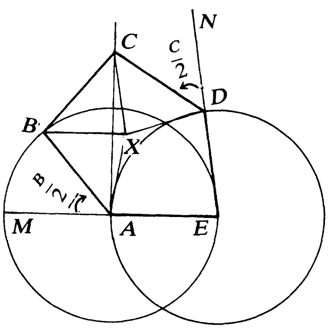
$B \sim 144°32'30''$ $\qquad D \sim 135°22'$

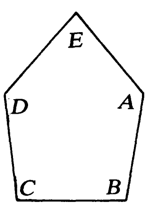
Illustrative tiling: (30)

TABLE III. Equilateral Pentagons Known to Tile. Types 8–9

Type 8.

$2A + B = 2\pi.$
$2D + C = 2\pi$





Construction

Let $A = P$ and $B = Q$ in FIGURE 11; then $C = R$, $D = S$, $E = T$. Adjust vertex R on line l until $2D + C = 2\pi$ (a unique solution). This is the unique pentagon of the family in FIGURE 11 satisfying $Q = R$, $P = S$. To see this, add construction lines to our pentagon $ABCDE$ as follows.

Extend EA to M and ED to N. Let X be the intersection of the bisectors of angles B and C; draw AX, BX, CX, DX. Then $\Delta XCD \simeq \Delta XCB \simeq \Delta XBA$ so $\sphericalangle XAB = C/2$ and $\sphericalangle XDC = B/2$. Now $\sphericalangle MAB = B/2$, and $\sphericalangle NDC = C/2$, which implies $\sphericalangle XAD = \sphericalangle XDA$. Thus $BX = CX$ and so angles B and C are equal, angles A and D are equal.

The explicit value for $\cos E$ can be obtained using the extended law of cosines in [20].

Angles

$E = \arccos \frac{\sqrt{13} - 3}{4}$

$\sim 81°18'$

$B = C = \pi - E \sim 98°42'$

$A = D = \frac{\pi}{2} + \frac{E}{2} \sim 130°39'$

$C + E = \pi$ and $B + E = \pi$ show this is also type 2.

Illustrative tilings

(31) and tilings listed for type 2 above.

Type 9.

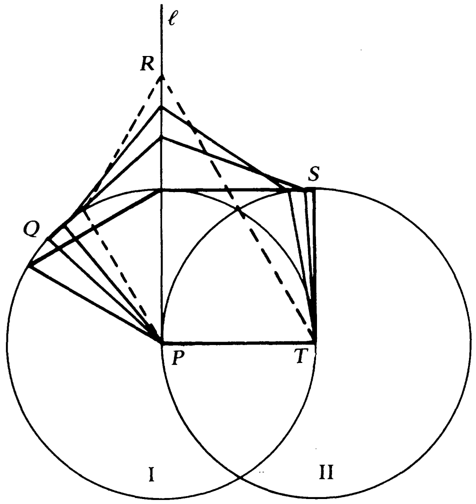
$2E + B = 2\pi$
$2D + C = 2\pi$

Impossible. The angle condition $2D + C = 2\pi$ puts it in the family in FIGURE 11, with $P = D$, $Q = C$, $E = T$. The condition $2E + B = 2\pi$ implies $E > \pi/2$, which contradicts the condition in FIGURE 11 that $E \leqq \pi/2$.

George Szekeres and Michael Hirschhorn of the University of New South Wales to conduct a week-long study group of Form 5 high school students on tilings by equilateral pentagons [16]. Both Rice and the Australian class discovered that a particular type 1 pentagon ($A = 140°$, $B = 60°$, $C = 160°$, $D = 80°$, $E = 100°$) could tile in curious ways (FIGURE 12). Both discovered tiling (40) composed of zigzag bands in which the pentagons can fit together in two distinct ways. The particularly beautiful tiling (41), having only rotational symmetry, was discovered by Hirschhorn. Since there are eleven ways in which the angles of this pentagon can be combined to make 360°, several other unusual tilings are also possible, including another design having only rotational symmetry. For this reason, Hirschhorn has dubbed it the "versa-tile."

Some questions

The general problem of determining all convex pentagons which can tile the plane remains unsolved. Is our list of 13 types which tile complete? We doubt it. Even though a complete solution of the problem appears to be difficult (certainly lengthy), perhaps the full answer to some special cases can be obtained. We have noted that the list of types of pentagons which can generate a tile-transitive tiling is complete (types 1 through 5). Are the only tiles capable of edge-to-edge tilings types 1, 2, 4, 5, 6, 7, 8 and 9? Is the list of equilateral pentagons which tile complete? I hope these questions will stimulate further activity on the problem.



The family of equilateral pentagons for which $2P + Q = 2\pi$ can be envisioned as generated by a flexible equilateral pentagon, hinged at the vertices, with side PT held fixed, while R rides up and down line l (causing vertices Q and S to ride along arcs on circles I and II respectively). The formal construction is as follows:

> **Choose angle $P > 2\pi/3$. Construct sides $PT = PQ$ by drawing circle I with center at P. Construct line l perpendicular to PT at P. Find vertex R on line l so $QR = QP$ (then $2P + Q = 2\pi$). Draw circle II with center T, radius PT. Find vertex S on circle II so that $RS = ST$ and $PQRST$ is convex.**

The conditions on the angles listed above follow easily from this construction.

Three pentagons in this family are known to tile the plane. In addition to the unique type 7 and unique type 8 (which is also type 2) listed in Table III, there is a unique type 1 pentagon: $P = 150°$, $Q = 60°$, $R = 150°$, $S = 90°$, $T = 90°$.

FIGURE 11.

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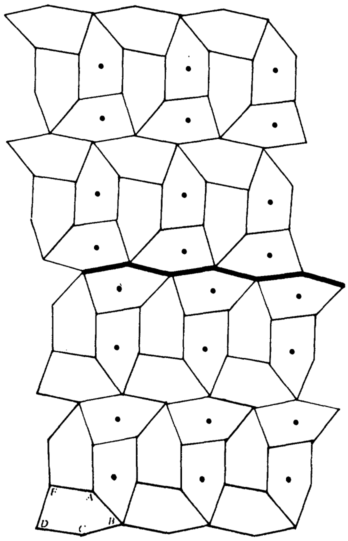

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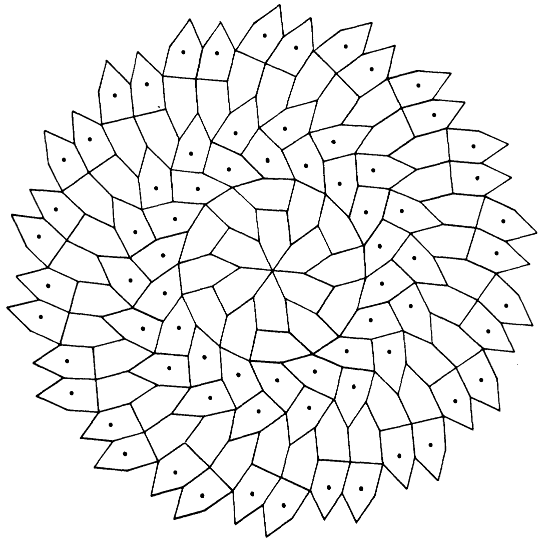

(41)

Non-transitive tilings by the equilateral pentagon $A=140°$, $B=60°$, $C=160°$, $D=80°$, $E=100°$. Pentagons fill zig-zag horizontal strips in two distinct ways in tiling (40); these strips fit together in innumerable ways to fill the plane. Tiling (41) by Michael Hirschhorn has only 6-fold rotational symmetry.

FIGURE 12

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Tic-Tac-Toe in n-Dimensions

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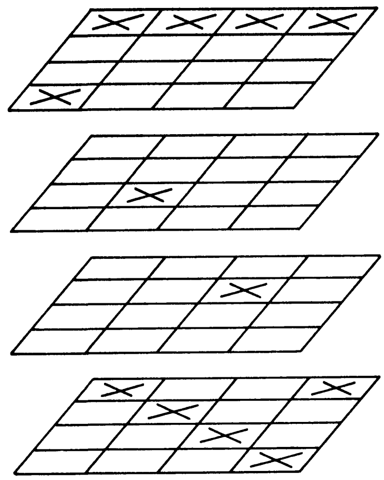
There are several three dimensional tic-tac-toe games currently being marketed. Most of those games are played on a cubical board having 4 cells on a side, which we shall call a $4\times4\times4$ game. Two players alternately choose a cell in the $4\times4\times4$ cube, the winner being the first player to obtain 4 cells lying in a row, called a **winning set.** Winning sets lie along the 48 orthogonal rows (rows parallel to one of the edges of the cube), the 24 diagonal rows, or the 4 main diagonals of the cube (see FIGURE 1), making 76 winning sets altogether.



Three typical winning sets in the $4\times4\times4$ game.

FIGURE 1.

In going from the traditional two dimensional 3×3 game of tic-tac-toe to a three dimensional game, there is good reason to add a fourth cell on each side, since it is easy to see that the first player has an easy win in the $3\times3\times3$ game if he takes the center cubical on his first move. Moreover, two players *cannot* play tic-tac-toe to a tie in the $3\times3\times3$ game even if they try! More precisely, whenever the cells in the $3\times3\times3$ cube are divided arbitrarily into two sets, then at least one of the two sets will contain 3 cells in a row. We shall prove this fact below.

Generalizing to n dimensions, it will be notationally convenient to consider an $m\times m\times\cdots\times m$ (n times) hypercube as having its cells centered at the points of an n-dimensional hypercube $C^n(m)$ of lattice points in euclidean n-space having m points on a side, e.g., $C^n(m)=\{(x_1,\ldots,x_n)\in Z^n: 1\leqq x_i\leqq m\}$, where Z denotes the set of integers. A **winning set** in $C^n(m)$ then consists of m points of $C^n(m)$ lying along a straight line. Tic-tac-toe is played in $C^n(m)$ by two players alternately selecting a point in $C^n(m)$, the winner being the first player to obtain a winning set. A **tie partition** of $C^n(m)$ is a partition of $C^n(m)$ into two sets which differ in cardinality by at most one, and neither of which contains a winning set. We discuss in this note the question of determining the values of n and m for which tie partitions of $C^n(m)$ exist.

We shall not consider the question of when winning strategies exist, except to mention the fact that whenever tie partitions do *not* exist, then a well-known theorem in game theory implies that the first player has a winning strategy. The proof goes as follows. Since the game cannot end in a tie, one of the two players has a winning strategy. If you assume that the second player has a winning strategy, let the first player make a random opening move and thereafter assume the role of the second player. At later stages of the game, the first player responds to the second player's moves as dictated by the second player's winning strategy, or moves at random if he has already made the dictated move. Since having made one extra move cannot possibly hurt at any stage, the first player is thereby led to a win, contradicting the assumed existence of a winning strategy for the second player.

Of course, the first player might have a winning strategy even when tie partitions exist, although we do not know of an example. (It is generally believed $C^3(4)$ is such an example.) We refer the reader to **[1]** for information on when winning strategies exist, and to **[2]** for a nice discussion of n-dimensional Tic-Tac-Toe and its variants.

We begin with an interesting calculation: *There are* $\frac{1}{2}((m+2)^n - m^n)$ *winning sets in* $C^n(m)$. Our first proof is an expansion of the comments made by Leo Moser in **[4]**. Note that m points of $C^n(m)$ which form a winning set can be ordered (there are two such orderings) $\alpha_1, \ldots, \alpha_m$, where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, $1 \leqq i \leqq m$, so that the sequence $\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj}$, $1 \leqq j \leqq n$, is one of the sequences $1, 2, \ldots, m$, or $m, m-1, \ldots, 1$, or a constant sequence $k, k, \ldots, k$, where $1 \leqq k \leqq m$. Let $\tilde{C}^n(m+2) = \{(x_1, \ldots, x_n) \in Z^n : 0 \leqq x_i \leqq m+1\}$, so that $C^n(m)$ is contained inside $\tilde{C}^n(m+2)$, and the "outer shell" $S = \tilde{C}^n(m+2) \setminus C^n(m)$, contains $(m+2)^n - m^n$ points. Now the straight line determined by the winning set $\{\alpha_1, \ldots, \alpha_m\}$ intersects S in a (unique) pair of points $\{x, y\}$ determined by requiring the sequence $x_j, \alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj}, y_j$, $1 \leqq j \leqq n$, to be one of the sequences $0, 1, 2, \ldots, m, m+1$, or $m+1, m, \ldots, 1, 0$, or a constant sequence $k, k, \ldots, k$, where $1 \leqq k \leqq m$. On the other hand, given any point $x = (x_1, \ldots, x_n) \in S$, we can determine the (unique) point $y = (y_1, \ldots, y_n) \in S$ so that the line going through x and y intersects $C^n(m)$ in a winning set. Indeed, $y_i = x_i$ when $1 \leqq x_i \leqq m$, $y_i = m+1$ when $x_i = 0$, and $y_i = 0$ when $x_i = m+1$. This shows that the winning sets in $C^n(m)$ are in one-to-one correspondence with the partition of S into pairs of points described above. This completes the proof.

An alternate proof of the formula can be given as follows. Let $W(n, m)$ denote the set of winning sets in $C^n(m)$, and let $W_r(n, m)$ be the subset of $W(n, m)$ consisting of those winning sets $\{\alpha_1, \ldots, \alpha_m\}$ where exactly $n - r$ of the coordinates of the α_i have fixed values, $1 \leqq r \leqq n$. For example, after introducing a suitable coordinate system into FIGURE 1, the three winning sets in $C^3(4)$ that are illustrated consist of

$$\{(1,1,4), (1,2,4), (1,3,4), (1,4,4)\} \in W_1(3,4),$$

$$\{(1,1,1), (2,2,1), (3,3,1), (4,4,1)\} \in W_2(3,4),$$

$$\{(1,4,1), (2,3,2), (3,2,3), (4,1,4)\} \in W_3(3,4).$$

Now $|W_r(m, n)| = \binom{n}{r} m^{n-r} 2^{r-1}$, $1 \leqq r \leqq n$, where $|A|$ denotes the number of elements of a set A. To see this, note that there are $\binom{n}{n-r} = \binom{n}{r}$ ways to choose the $n - r$ coordinates which are to have a fixed value for each point in the winning set, and m choices of the constant value for each of these coordinates, yielding $\binom{n}{r} m^{n-r}$ such choices. We then pick one of the remaining r coordinates which vary, and order the points in the winning set so that this coordinate varies in the order $1, 2, \ldots, m$. Then the remaining $r - 1$ coordinates either vary in the order $1, 2, \ldots, m$ or in the opposite order $m, m-1, \ldots, 1$, so that we obtain 2^{r-1} such choices. Hence, $|W_r(n, m)| = \binom{n}{r} m^{n-r} 2^{r-1}$ and

$$|W(n,m)| = \sum_{r=1}^{n} |W_r(n,m)| = \sum_{r=1}^{n} \binom{n}{r} m^{n-r} 2^{r-1} = \tfrac{1}{2}((m+2)^n - m^n).$$

The situation for tie partitions in dimensions $n \leqq 3$ is well known: tie partitions of $C^n(m)$ exist when $m > n$, but tie partitons of $C^n(n)$ do not exist when $n = 1, 2,$ or 3. Indeed a stronger statement can be made: *If* $n \leqq 3$ *and* $C^n(n) = A \cup B$, *then either* A *or* B *contains a winning set.*

This statement is trivial for $n = 1$ or 2. The following proof for $n = 3$ exploits the symmetries of the cube and reduces the task of examining all partitions of $C^3(3)$ to a quick check of three cases. First,

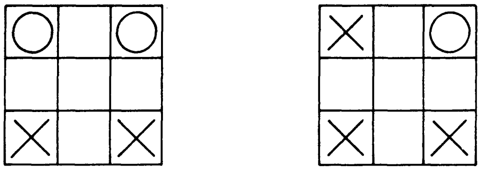


FIGURE 2.

note that up to symmetries of the square and an interchange of the two sets in the partition, there are only two partitions of the corner points of $C^2(3)$ which can be extended to tie partitions of $C^2(3)$. These two corner partitions are shown in FIGURE 2, and will be termed **admissible.**

We place the first partition in FIGURE 2 in any one of the six outside faces of $C^3(3)$, and extend the partition to the center point $(2,2,2)$ of $C^3(3)$ in one of the two possible ways. As indicated in FIGURE 3, an attempt to further extend the partition avoiding winning sets leads to an inadmissible corner partition in the outside face. Placing the center point in the other set leads, of course, to an entirely symmetric situation.

On the other hand, if we place the second partition in FIGURE 2 in an outside face, we have two cases to consider, depending on how we choose to extend the partition to the center point. Now any attempt to extend the partition avoiding winning sets leads to an untenable position as indicated by

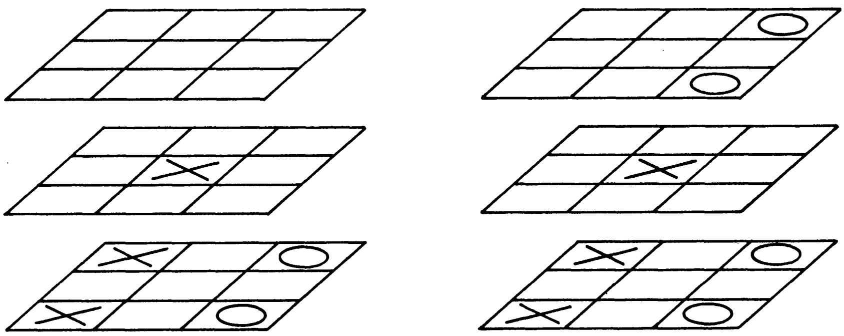


FIGURE 3.

the symbol ⊗ in FIGURES 4 and 5. In FIGURES 4 and 5 we indicate only enough forced moves to lead to an untenable position. This completes the verification of the proposition.

An interesting situation arises when the center point is removed from $C^3(3)$. FIGURE 6 shows a partition of $C^3(3)\setminus\{(2,2,2)\}$ into two sets neither of which contains a winning set. It can be shown that such a partition is unique up to symmetries of the cube and an interchange of the two sets in the

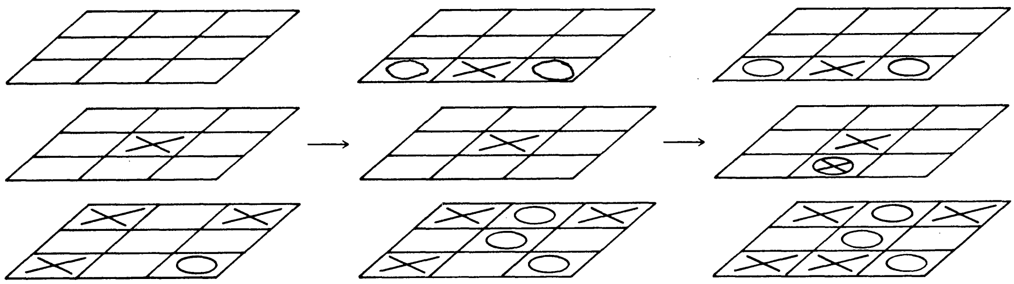


FIGURE 4.

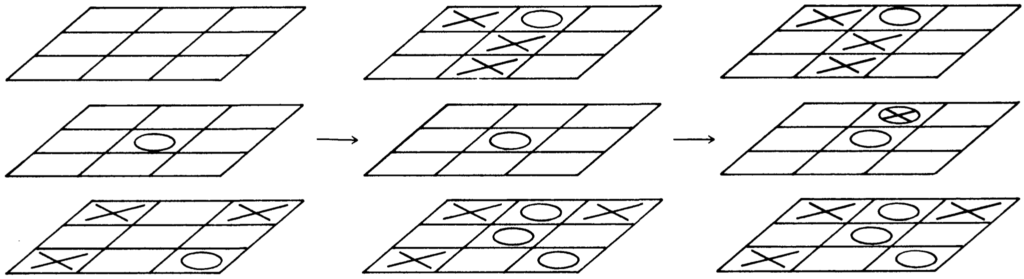


FIGURE 5.

partition. Moreover, note that the two sets in the partition differ in cardinality by *two*, (so that tie partitions of $C^3(3)\setminus\{(2,2,2)\}$ do not exist).

FIGURE 7 shows a partition of $C^3(4)$. Extending the partition periodically to all of Z^3 yields tie partitions of $C^3(n)$, $n \geqq 4$. In fact, suppose we define a winning set of length m in Z^n to be a winning set in any translate of $C^n(m)$. It is easy to verify that neither set in the periodic extension of the partition in FIGURE 7 to all of Z^3 contains a winning set of length 4. Moreover, "horizontal" sections of the partition do not contain a winning set of length 3. We now can state precisely the open question alluded to earlier: Does there exist a hypercube $C^n(m)$ such that the first player has a winning strategy even though tie partitions of $C^n(m)$ exist? In particular, is $C^3(4)$ such an example?

In contrast to the result for $C^n(n)$ when $n \leqq 3$, it has been shown by A. W. Hales that tie partitions *do* exist for $C^4(4)$. Hales and Jewett have shown in **[3]** that whenever $m \geqq n+1$, $C^n(m)$ can be partitioned into two sets neither of which contains a winning set (although the two sets they use differ in cardinality by more than one). In fact, tie partitions of $C^n(m)$ do exist whenever $n \geqq 4$, and $m \geqq n$. The rather involved proof of this latter result is contained in **[7]**. On the other hand, Hales and Jewett have shown in **[3]** that given any m, there is an $n > m$ such that whenever $C^n(m) = A \cup B$, then either A or B contains a winning set.

As an extension of these concepts, suppose we consider the problem of partitioning Z^n into two sets in such a way as to minimize (over all such partitions) the length of any winning set contained in either set in the partition. More precisely, define $\rho(Z^n)$ to be the largest positive integer k such that whenever $Z^n = A \cup B$, then either A or B contains a winning set of length k. The results of this note show that $\rho(Z^n) = n$ for $n = 1,2,3$. In contrast to this result, the author has shown (see **[6]**, **[7]**) that $\rho(Z^n) \leqq n-1$ for $n \geqq 4$, $\rho(Z^n) \leqq n-2$ for $n \geqq 8$, and while $\rho(Z^4) = 3$, the exact value of $\rho(Z^n)$ when

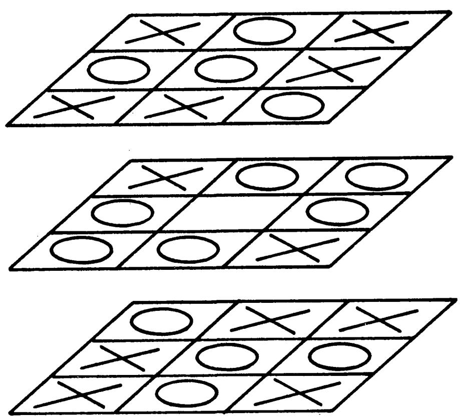


FIGURE 6.

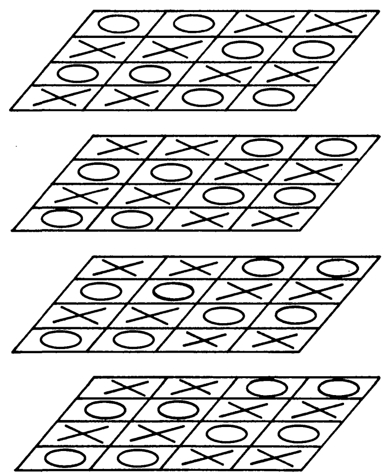


FIGURE 7.

$n \geqq 5$ remains open. A result of Hales and Jewett referred to above shows that $\lim_{n\to\infty} \rho(Z^n) = \infty$. It would be interesting to determine the asymptotic nature of this growth. In particular, is $\lim_{n\to\infty} \rho(n)/n = 1$?

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Stretch: A Geoboard Game

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Stretch is played on a geoboard with one rubber band. A geoboard is a flat piece of wood with small nails hammered in at lattice points; a one foot square of plywood with nails making one inch squares works nicely. (A good source of interesting geoboard questions is **[3]**.) Two players start with a convex polygon having at least one interior nail, such as the one in Figure 1a. Players move by modifying the figure subject to the following rules:

(1) The modified figure must be convex.

(2) The number of nails in the interior of the modified figure must be the same as the number of nails in the interior of the original figure. (In Figure 1 there are 2 interior nails.)

(3) The number of nails on the boundary of the modified figure must be greater than the number of nails on the boundary of the preceding figure.

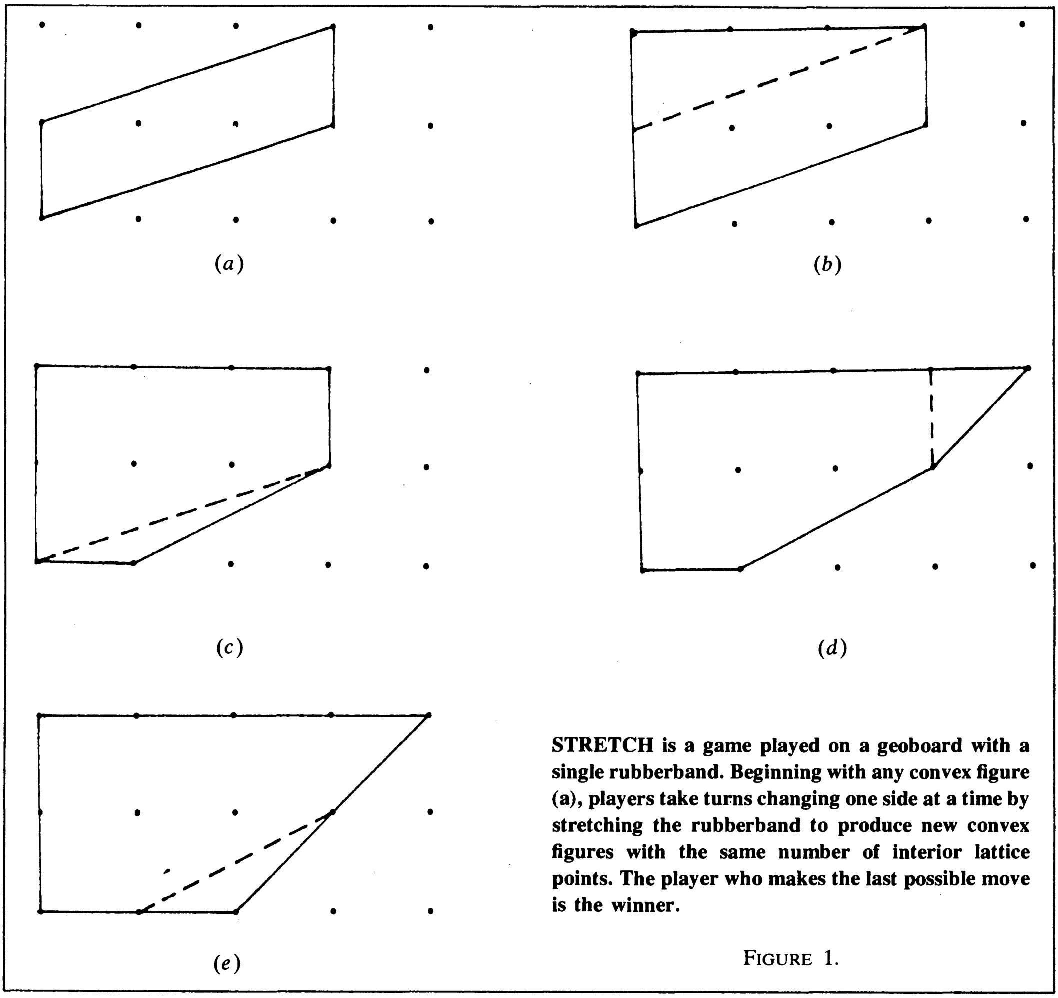
(4) At most one side of the current figure may be disturbed. (A side may be extended without being disturbed.)

Play continues until no legal move is possible. The rules assume that the geoboard is large enough to accommodate any possible legal move. The last player to make a legal move is the winner.

Figure 1b shows a legal modification of Figure 1a. Note that the left vertical side has been extended. The number of boundary nails has increased from 4 to 7 and there are still 2 interior nails. The entire sequence in Figure 1 shows a typical game of Stretch. The second player is the winner, even though he is not a very smart player.

One question that arises immediately is this: Will the game necessarily come to an end? (This is the same as asking whether Stretch is a game in the sense of **[1]** for any initial figure.) If we start with a figure having no interior nails, or if we do not insist on convexity, the game will go on forever. In other cases, however, the game must come to an end:

Theorem. *Suppose that a convex figure on a geoboard has B boundary nails and I interior nails, with $I > 0$. Then $B \leqq 9I$.*



STRETCH is a game played on a geoboard with a single rubberband. Beginning with any convex figure (a), players take turns changing one side at a time by stretching the rubberband to produce new convex figures with the same number of interior lattice points. The player who makes the last possible move is the winner.

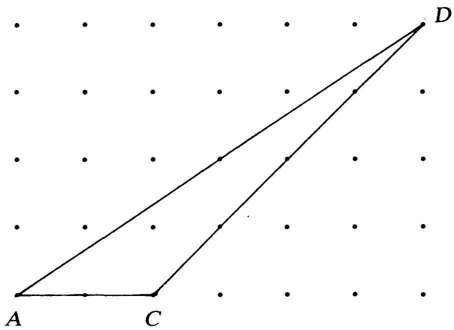
FIGURE 1.

This is a thorem about convex polygons whose vertices are lattice points, but the language of geoboards and nails is descriptive, and quite adequate for our purposes. We shall outline a proof of the theorem based on a series of lemmas. The reader is encouraged at this point to get hold of a geoboard or, in case no geoboard is available, several sheets of graph paper.

We use the term **geoboard figure** to designate a simple closed polygon whose vertices are lattice points. Our proof is based on the following fascinating result, known as Pick's Theorem: *Suppose a geoboard figure has area A, with B boundary nails and I interior nails. Then* $A = B/2 + I - 1$. (For the parallelogram in FIGURE 1a, $B = 4$ and $I = 2$; and indeed $A = 4/2 + 2 - 1$.) For more information on Pick's Theorem, see [2] and its references.

In order to compute the number of boundary nails for a figure it suffices to be able to compute the number of spaces between nails on a line segment connecting two nails. For if a segment has t such nails, including the endpoints, then there are $s = t - 1$ spaces between nails on the segment. So if a figure has n sides, and if the number of spaces along the kth side is s_k, then $B = s_1 + s_2 + \cdots + s_n$. (In FIGURE 2, $s_1 = 2$, $s_2 = 4$ and $s_3 = 2$ for the segments AD, DC and CA, respectively; thus $B = 2 + 4 + 2 = 8$. Hence $A = 4 + 1 - 1 = 4$.) It is convenient to coordinatize the lattice points using integers and compute the number of spaces between two nails in terms of their coordinates. We state this carefully:

LEMMA 1. *Let $A(0,0)$ and $D(a,b)$ be coordinatized nails such that the segment AD has s spaces. Then $s =$ g.c.d. (a,b), the greatest common divisor of a and b.*



PICK'S THEOREM says that the area of a lattice triangle such as ACD is one less than half the number of boundary lattice points plus the number of interior lattice points. Hence in this Figure, the area of ACD is $(8/2)+1-1=4$.

FIGURE 2.

Proof. Note first, as an example, that in FIGURE 2 we have $a = 6$ and $b = 4$ for the segment AD. Happily, g.c.d. $(6,4)=2$ and $s = 2$. More generally, let d denote g.c.d. (a,b) and write $a = du$, $b = dv$. Then the d points $(u,v),(2u,2v),\ldots,(du,dv)$ lie on the segment AD; hence $d \leqq s$. On the other hand the segment AD projects onto the x-axis to subdivide the interval $[0,a]$ into s segments of equal length; and on the y-axis the interval between $(0,0)$ and $(0,b)$ breaks into s segments of equal length. Hence s is a common divisor of a and b. No common divisor exceeds the greatest common divisor, so $s \leqq d$. We conclude that $s =$ g.c.d. (a,b), completing the proof.

It is easy to shift coordinates and see that the number of spaces in the segment between any two points, say $P(a,b)$ and $Q(c,d)$, is g.c.d. $(a-c, b-d)$. So, given the coordinates of the vertices of a figure, we have a method for computing the number of boundary nails.

Next we wish to be able to transform a given figure to another one that has the same number of boundary nails and interior nails. The appropriate transformations are the **unimodular** transformations given by $\tau(x,y)=(\alpha x+\beta y, \gamma x+\delta y)$, where $\alpha,\beta,\gamma,\delta$ are integers such that the determinant $\alpha\delta-\beta\gamma$ equals ± 1. Such a transformation takes lattice points to lattice points, and therefore moves a geoboard figure to another geoboard figure.

LEMMA 2. *Let X and X' be geoboard figures having B and B' boundary nails, I and I' interior nails, and areas A and A', respectively. If τ is a unimodular transformation such that $\tau(X)=X'$, then $B = B'$, $I = I'$ and $A = A'$.*

Proof. First note that if a linear tranformation τ has determinant Δ, then Area $(\tau(X)) = |\Delta|$. Area (X) for any region X. (For our purposes it suffices to check this for triangles.) In our case $|\Delta|=1$, so $A = A'$. Let $\tau(x,y)=(\alpha x+\beta y, \gamma x+\delta y)$. We argue that for any integers a and b, g.c.d. $(a,b)=$ g.c.d. $(\alpha a+\beta b, \gamma a+\delta b)$. For suppose d is a common divisor of a and b. Then clearly d is a common divisor of $\alpha a+\beta b$ and $\gamma a+\delta b$. Conversely, suppose that c is a common divisor of $\alpha a+\beta b$ and $\gamma a+\delta b$. Then $\alpha a+\beta b = cu$ and $\gamma a+\delta b = cv$ for some u, v. Fix all symbols except a and b and solve for a and b, obtaining $a = c((\delta u-\beta v)/(\alpha\delta-\beta\gamma))$ and $b = c((\gamma u-\alpha v)/(\alpha\delta-\beta\gamma))$. Since $\alpha\delta-\beta\gamma = \pm 1$, this says that c is a common diviosr of a and b. Thus these two pairs of integers have the same common divisors.

Using Lemma 1 we now see that $B = B'$, for B and B' are computed as sums of g.c.d.'s which must be equal term by term. Since $A = A'$, Pick's Theorem gives $\frac{1}{2}B+I-1=\frac{1}{2}B'+I'-1$; and since $B = B'$, it follows that $I = I'$. This proves Lemma 2.

We notice next that we may move any figure to one having a side on the x-axis and a vertex at the origin without changing B, A and I.

LEMMA 3. *Let a and b be integers and let $d =$ g.c.d. (a,b). Then there is a unimodular transformation τ such that $\tau(a,b)=(d,0)$.*

Proof. Write $a = du$, $b = dv$ with g.c.d. $(u,v)=1$ and choose integers α and β such that $\alpha u+\beta v = 1$; let $\gamma = -v$, $\delta = u$. Then τ defined by $\tau(x,y)=(\alpha x+\beta y, \gamma x+\delta y)$ takes (a,b) to $(d,0)$; and $\alpha\delta-\beta\gamma = \alpha u+\beta v = 1$, so τ is unimodular.

Now any linear transformation takes convex regions to convex regions (since line segments go to line segments); and if a figure has n-sides and is transformed to another figure under a 1–1 linear transformation, the image has n sides. Thus a unimodular transformation must take a convex n-gon to a convex n-gon. Unimodular transformations correspond precisely to those matrices having integer entries whose inverses also have integer entries.

LEMMA 4. *If a triangle on a geoboard has B boundary nails and I interior nails, with* $I > 0$, *then* $B \leqq 2I + 7$. *If* $I \geqq 2$, $B \leqq 2I + 6$.

Proof. Using Lemmas 2 and 3, we may assume that our triangle has vertices $(0,0)$, $(e,0)$ and (a,b), where e, a and b are positive integers; by proper choice of vertices we may assume $e \geqq \text{g.c.d.}(a,b)$ and $e \geqq \text{g.c.d.}(a-e,b)$. If $e \leqq 3$, then $B \leqq 9 \leqq 2I + 7$ ($2I + 6$ if $I \geqq 2$) so there is nothing to prove. Hence we may take $e \geqq 4$. Since $I > 0$ we have that $b \geqq 2$. We shall show that $B \leqq 2I + 6$ under the assumption that $e \geqq 4$.

We have a triangle, call it T, with $B = e + \text{g.c.d.}(a,b) + \text{g.c.d.}(a-e,b)$. Consider the triangle T_1 (see FIGURE 3) with vertices $(0,0), (0,b)$ and $(e,0)$. Then T_1 has $B_1 = e + b + \text{g.c.d.}(e,b)$ boundary nails. One can show that $b + \text{g.c.d.}(e,b) \geqq \text{g.c.d.}(a,b) + \text{g.c.d.}(a-e,b)$. (Equality holds precisely when one of the two terms on the right equals b.) Hence $B_1 \geqq B$. Since both T and T_1 have area $\frac{1}{2}be$, Pick's Theorem gives $\frac{1}{2}B_1 + I_1 - 1 = \frac{1}{2}B + I - 1$, where T_1 has I_1 interior nails. Whence $I_1 \leqq I$. Since $e \geqq 3$ and $b \geqq 2$, it is clear that $I_1 > 0$. We now prove that $B_1 \leqq 2I_1 + 6$. Since $B_1 \geqq B$ and $I_1 \leqq I$, this will suffice.

We compute B_1 and I_1.

$$B_1 = b + e + \text{g.c.d.}(b,e);$$

and

$$I_1 = \tfrac{1}{2}[(b-1)(e-1) - \text{g.c.d.}(e,b) + 1]$$

comes from counting the nails interior to the rectangle $(0,0)$, $(e,0)$, (e,b), $(0,b)$ except those on one of the diagonals. Then $B_1 \leqq 2I_1 + 6$ translates into: $2[b + e + \text{g.c.d.}(b,e)] \leqq be + 8$, But $2[b + e + \text{g.c.d.}(b,e)] \leqq 2e + 4b$ since $\text{g.c.d.}(e,b) \leqq b$; and $2e + 4b \leqq be + 8$ follows from $(b-2)(e-4) \geqq 0$. Thus in case $e \geqq 4$ we have $B_1 \leqq 2I_1 + 6$, proving the lemma.

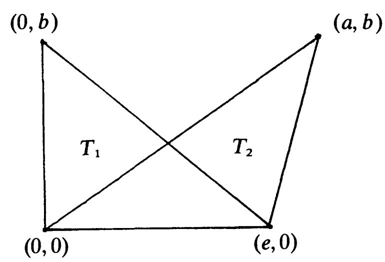


FIGURE 3.

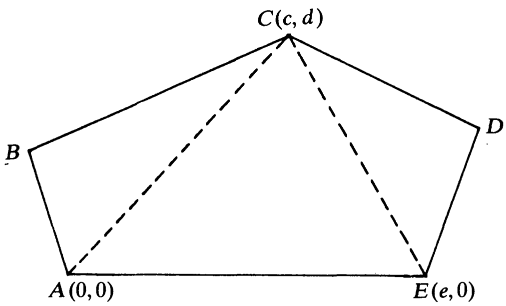


FIGURE 4.

The bound $B = 2I + 6$ is attained for the triangle with vertices $(0,0)$, $(0,2)$, $(2I+2,0)$. It is interesting to note that if a triangle has exactly 1 interior nail, it must have 3, 4, 6, 8 or 9 boundary nails. The theorem gives 9 as the maximum possible number. Using the notation of the triangle T above, Pick's Theorem gives $eb = e + \text{g.c.d.}(a,b) + \text{g.c.d.}(a-e,b) = B$. Thus B is a sum of three of its divisors. But this is impossible for $B = 5$ or 7. It is possible for the values 3, 4, 6, 8 and 9 and these expressions for B lead to examples having these numbers of boundary nails. See **[4]** for an alternative proof of this special case. (The question of whether a triangle can have 5 boundary nails and 1 interior nail started this investigation. It was posed to the author several years ago by a seventh grade student named Kevin James.)

LEMMA 5. *Let ABCDE be a geoboard pentagon. If the triangle ACE has no interior nails, then one of the segments AC or EC must contain an interior nail. If the segment AE has at least one interior nail, then each of the two segments AC and EC has an interior nail.*

Proof. By Lemmas 2 and 3 we may be sure $A = (0,0)$, $C = (c, d)$ and $E = (e, 0)$; see FIGURE 4. Now suppose ACE has no interior nails, and neither AC nor CE has an interior nail. Use Pick's Theorem to compute the area of ACE, getting $\frac{1}{2}de = \frac{1}{2}(e+2) - 1 = \frac{1}{2}e$, which gives $d = 1$. But this is impossible since $ABCDE$ is a (convex) pentagon.

Now for the second part of the lemma, suppose $e \geqq 2$ and that AC has no interior nail. Then as before,

$$\tfrac{1}{2}de = \tfrac{1}{2}[e + \text{g.c.d.}(c-e, d) + 1] - 1.$$

Since $\text{g.c.d.}(c-e, d) \leqq d$, this gives $\frac{1}{2}de \leqq \frac{1}{2}(e + d + 1) - 1$, which implies $(d-1)(e-1) \leqq 0$. But $e \geqq 2$ by assumption and $d \geqq 2$ by convexity, so this is impossible. It is similarly impossible that CE has no interior nail. This completes the proof.

As a corollary to this lemma, we note that every convex pentagon has at least one interior nail. Next, we need to know that the number of sides of a figure cannot be large compared to the number of interior nails. Our next (and final) lemma provides a bound. In this lemma, we let $[x]$ denote the greatest integer in x.

LEMMA 6. *If a convex n-gon has I interior nails, then* $I \geqq [(n-3)/2]$. *For* $n \geqq 7$, *we have* $I \geqq [(n-2)/2]$.

Proof. If $n = 3$ or 4, then $[(n-3)/2] = 0 \leqq I$. If $n = 5$ or 6, we have $[(n-3)/2] = 1$; and since a convex pentagon has an interior nail, $I \geqq 1$ in these cases. So let $n \geqq 7$ and let k be the positive integer satisfying $2k + 5 \leqq n < 2k + 7$. Then $[(n-2)/2] = k + 1$ if $2k + 5 = n$; and $[(n-2)/2] = k + 2$ if $2k + 6 = n$.

Case 1. Suppose $2k + 5 = n$. We first note that either the quadrilateral Q_0 in FIGURE 5 has an interior nail or the segment A_1A_4 has an interior nail. This follows from Lemma 5, applied to the pentagon $A_1A_2A_3A_4A_5$, with A_3A_4 playing the role of AE in the lemma. Next we claim that each of the quadrilaterals $Q_1, \ldots, Q_k$ has an interior nail. To see this for Q_1, consider the pentagon $A_1A_3A_4A_5A_6$. If the triangle $A_4A_5A_1$ has an interior nail, so does Q_1. If not, then by Lemma 5 one of the segments A_1A_4 or A_1A_5 has an interior nail. If A_1A_5 has one, it is also interior to Q_1; so suppose A_1A_4 has an interior nail and apply Lemma 5 to the pentagon $A_4A_5A_6A_7A_1$. If the triangle $A_4A_6A_1$ has an interior nail we are through; if not, then A_4A_6 has an interior nail which is interior to Q_1. Thus

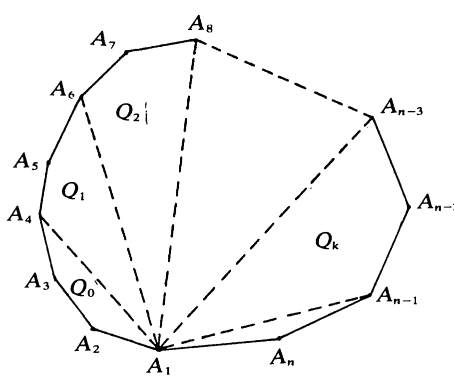


FIGURE 5.

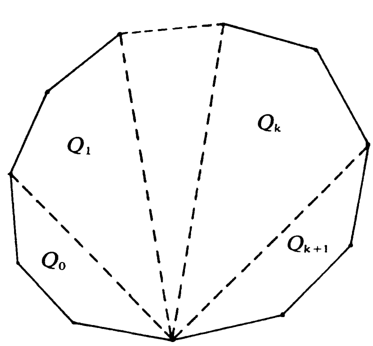


FIGURE 6.

in any case Q_1 has an interior nail. Since each Q_i, $i \geqq 2$, has two sides interior to the figure, the same argument gives each an interior nail. This accounts for $k+1=[(n-2)/2]$ interior nails, so $I \geqq [(n-2)/2]$ in this case.

Case 2. Suppose $2k+6=n$. We can now label our n-gon as in FIGURE 6. As we saw above, each $Q_1, \ldots, Q_k$ has an interior nail. Moreover, using Lemma 5 again, each of Q_0, Q_{k+1} has a nail in its interior or on its boundary that is interior to the figure. Hence we count at least $k+2=[(n-2)/2]$ interior nails, so $I \geqq [(n-2)/2]$ in this case. This proves Lemma 6.

We are now ready to argue that $B \leqq 9I$ for any convex geoboard figure with n sides. The proof is by induction on n. We have proved this for $n=3$ (Lemma 4) and shall wave our hands at the cases $n=4,5,6$. (The details consist of persistent use of Lemmas 1–5; the author will send the details to anyone who is interested in them.) So we assume that $n \geqq 7$ and proceed inductively.

First note that if $n=B$, then $B=n \leqq 2[(n-2)/2]+3$; so by Lemma 6, $B \leqq 2I+3 < 9I$. Thus we may assume that at least one side of the figure has an interior nail; by Lemmas 2 and 3 we may put our figure in the form pictured in FIGURE 7, where $e \geqq 2$.

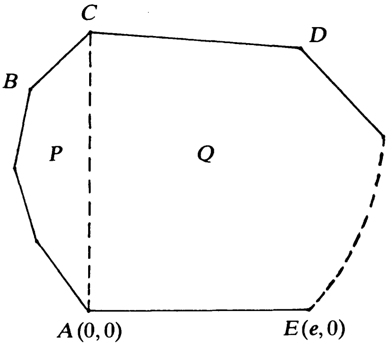


FIGURE 7.

The pentagon P surely has an interior nail; and applying Lemma 5 to the pentagon $ABCDE$, we conclude that the quadrilateral $ACDE$ has an interior nail. Thus our figure is made up of two smaller convex figures, each having an interior nail and each having fewer than n sides. So by induction we have $B_1 \leqq 9I_1$, and $B_2 \leqq 9I_2$, where P and Q have B_1 and B_2 boundary nails and I_1 and I_2 interior nails, respectively. Now $B \leqq B_1+B_2$ and $I_1+I_2 \leqq I$, so $B \leqq 9I_1+9I_2 \leqq 9I$. This completes the proof of the theorem, up to a few troublesome details.

This is not a very good bound. Since there is a triangle having $B=9$ and $I=1$ it is certainly the best bound of the form $B \leqq kI$. However, one can see by starting with a given figure and inflating it by a factor of n (that is, applying the transformation $\tau(x,y)=(nx,ny)$) that the number of boundary nails of the inflated figure is n times that of the original. But the area grows by a factor of n^2, so by Pick's Theorem the number of interior nails grows quadratically with n. So eventually B is very small compared to I. Examples have led to the following reasonable conjecture: *If a convex geoboard figure with n sides has B boundary nails and I interior nails, with $I>0$, then $B \leqq 2I+10-n$.*

If B is as small as possible, that is, $B=n$, then the conjecture leads to $n \leqq I+5$. This can be proved using Lemma 6 and an argument using induction on the area of the figure. (Pick's Theorem allows induction on area, since it gives all areas in the form $m/2$, m a positive integer.) For $n=4$ we can always produce a rectangle having $B=2I+6$ for a given value of I: take the vertices to be $(0,0)$ $(0,2)$, $(I+1,2)$, $(I+1,0)$. This bound, $2I+10-n$, is probably much too large for large values of n.

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An Infinite Class of Deltahedra

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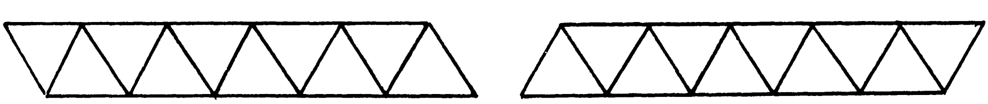
The appropriate designation **deltahedra** (Δ-hedra) has been proposed by Cundy [**1**] for polyhedra all faces of which are equilateral triangles. Three of the regular (Platonic) solids — the tetrahedron, octahedron, and icosahedron — are convex deltahedra with 4, 8, and 20 equilateral triangular faces, respectively. Models of deltahedra can be made conveniently by using No. 10 rubber bands to join the 3-inch flanged equilateral triangle cardboard panels invented by Seattle architect, Fred Bassetti [**2**]. Kits of these "Poly-O panels" can be purchased from Book-Lab, Inc. [**3**]. Or, home-made panels can be cut from railroad board in the manner clearly described by Stewart [**4**], Pritchett [**5**], and Woolaver [**6**].

Three convex deltahedra can be formed by placing together the bases of two congruent pyramids: the 6-faced triangular dipyramid or ditetrahedron, the 8-faced square dipyramid or octahedron, and the 10-faced pentagonal dipyramid. Six equilateral triangles can be assembled into a regular hexagon. Two such hexagons joined around their perimeters form a degenerate dipyramid. If this is separated into two parts along two coincident long diagonals, the two parts can be opened up by pressing inward on the ends of the diagonals. After one of the opened parts is rotated through 90°, the two parts can be rejoined to form a 12-faced Siamese dodecahedron which is convex.

There are only eight convex deltahedra [**7**]. Completing the set are the 14-faced triaugmented triangular prism, formed by attaching square pyramids to the square lateral faces of a triangular prism; and the 16-faced dicapped square antiprism, formed by attaching square pyramids to the two bases of a square antiprism. Thus the eight convex deltahedra are formed from 4, 6, 8, 10, 12, 14, 16 and 20 equilateral triangles. There are no convex deltahedra with 18 faces, nor any with more than 20 faces.

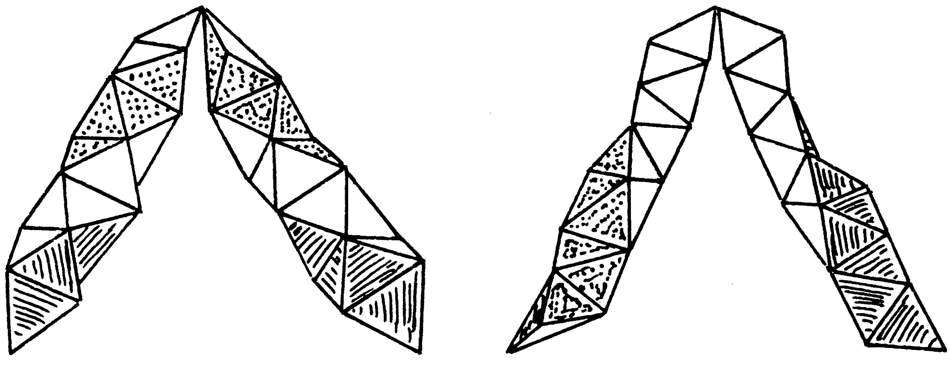
Once convexity is abandoned, the possibilities of forming deltahedra are endless. Members of one particularly interesting and attractive type are made by attaching regular tetrahedra to some or all of the faces of other deltahedra. The deltahedra thus made by augmentation are pseudo-stellated, since they are not the same as the solids produced by extending the planes of the faces of the basic polyhedron, except in the case of the augmented octahedron which is Kepler's famous *stella octangula.* A particularly attractive model is the pseudo-stellated icosahedron.

Models of an infinite subclass of deltahedra, the spiral (twisted) type, can be constructed from strips of equilateral triangles (FIGURE 1) flexing about the common sides. Any desired lengths of these strips can be formed easily by the cardboard panel-rubber band method. Models of spiral deltahedra can be started by joining the sides of the left-most angles of three strips to form a trihedral angle. The construction proceeds by joining sides of triangles of adjacent strips that have an end in common. (Differently colored strips emphasize the spiral nature of the deltahedron.) The three strips may be chosen from the H- and L-patterns in four ways, namely: *HHH*, *LLL*, *HHL*, and *LLH*. The *HHH* form appears to twist to the right (see FIGURE 2), and the *LLL* form appears to twist to the left. However, in general, each form consists of a pile of *regular* octahedra capped top and bottom with a



***H*-strip** ***L*-strip**

FIGURE 1.



LLL **Twist left** — *HHH* **Twist right** — *HHL* **Twist right** — *LLH* **Twist left**

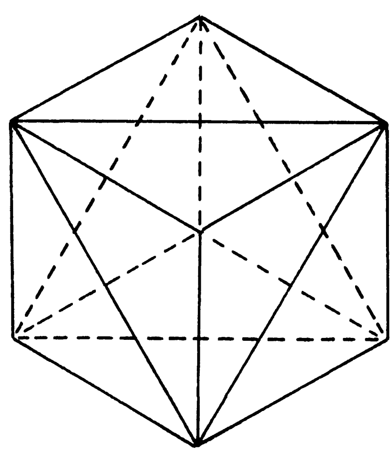
FIGURE 2.

regular tetrahedron. If each strip contains the same odd number of triangles, the model has an open face at one end. This could be closed to complete an octahedron by adding an additional triangle to one strip. If each strip contains $2n$ triangles, then the spiral polyhedron is composed of 2 regular tetrahedra and $(n-1)$ regular octahedra.

The dihedral angles of the regular tetrahedron and regular octahedron are arccos (1/3) or 70°32′ and 109°28′, respectively, and therefore are supplementary **[8]**. Consequently, at each end of the spiral polyhedron three faces of the tetrahedron and three faces of the adjacent octahedron form three plane rhombi. Thus, the *HHH* and *LLL* forms are **pseudo-deltahedra,** except when $n = 1$; then they are deltahedra and convex. When $n = 2$, they are rhombic hexahedra. When $n > 2$, the dihedral angles of a pseudo-deltahedron are variously 70°32′, 109°28′, and the re-entrant 218°56′.

The volume of a regular tetrahedron with edge e is $\sqrt{2}\,e^3/12$. The ratio of the volume of a regular octahedron and a regular tetrahedron each with edge e is 4 : 1 **[8]**. Therefore, the volume of a spiral pseudo-deltahedron is $(\sqrt{2}\,e^3/12)\,[2+4(n-1)]$ or $\sqrt{2}(2n-1)e^3/6$.

The join of the remote vertices of the terminal tetrahedra is an axis of the pseudo-deltahedron. The projection of the solid with $n > 1$ on a plane perpendicular to this axis is shown in FIGURE 3. There $AC = e = 2AB$, so $OA = e/\sqrt{3}$. Thus, the psuedo-deltahedron can be passed through a circular cylinder of radius $e/\sqrt{3}$.



Projection of *HHH* and *LLL* forms

FIGURE 3.

A spiral pseudo-deltahedron can be converted into a deltahedron by removing a triangle from each end of two of the strips and completing octahedral surfaces with the triangles at the ends of the longer strip. This deltahedron is a pile of octahedra. On a pile of six octahedra, each strip twists through 360°. Now, the inradius of a regular octahedron is $e/\sqrt{6}$; and in FIGURE 3, $OD = OA/2 = e/2\sqrt{3}$. Hence, in a pile of k octahedra a right circular cylinder of radius $e/2\sqrt{3}$ and length $2ke/\sqrt{6}$ can be inscribed.

The *HHL* and *LLH* forms are mirror images of each other, and are *bona fide* deltahedra. The *HHL* form twists to the right; and the *LLH* form twists to the left. When each strip contains 12 triangles, the twist is through 360°. If each strip contains the same odd number of triangles, the model has an open face at one end which may be closed by adding an additional triangle to one strip. If each strip contains $2n$ triangles, the spiral deltahedron consists of a pile of $(n-1)$ concave octahedra capped at each end with a regular tetrahedron. (A concave octahedron with equilateral triangular faces consists of three regular tetrahedra with a common edge.) It follows that the volume of a spiral deltahedron is given by $(\sqrt{2}e^3/12)[2+3(n-1)]$ or $\sqrt{2}e^3(3n-1)/12$. The dihedral angles of spiral deltahedra are variously 70°32′, 141°4′, and the reentrant 211°36′.

These spiral deltahedra may also be considered to be a special piling of regular tetrahedra or of regular tetrahedra and triangular dipyramids. A fuller appreciation of these deltahedra can be gained by actually constructing models.

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Cleopatra's Pyramid

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Nearly everyone has seen, if not solved, the geometrical puzzle Instant Insanity. This puzzle consists of four cubes, each face of which is one of four colors. The object of the puzzle is to assemble the cubes in a stack four high so that all four colors are visible on each side of the stack. For each cube there are three axes of symmetry which are perpendicular to the faces. Hence, the first cube can be positioned in 3 distinct ways, while for each of the remaining cubes there are 3 choices for the vertical axis, 2 choices for the orientation (up or down) of the vertical axis, and 4 rotations about the vertical axis. Hence, there are $3\cdot(3\cdot 2\cdot 4)^3 = 2(12)^4$ different arrangements **[1, 2, 3, 4]**. But there is only *one* solution.

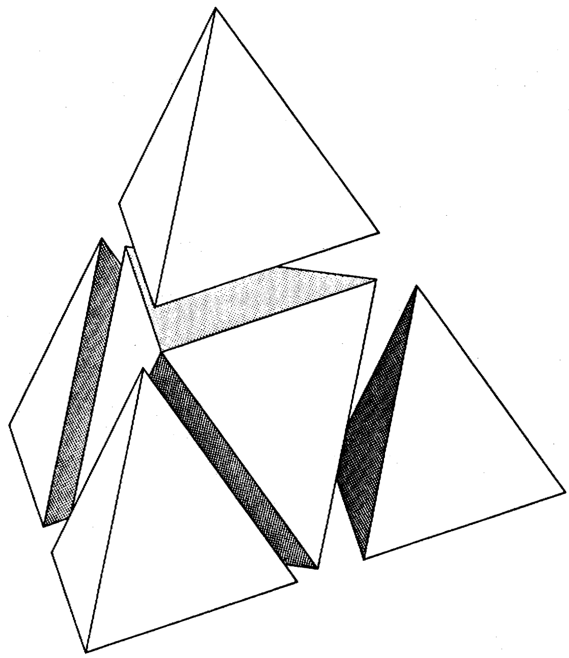


FIGURE 1.

In this paper, we consider a tetrahedral analog of Instant Insanity. The five pieces consist of four unit regular tetrahedra and one unit regular octahedron; see FIGURE 1. The faces are colored using four colors. The five pieces are easily arranged to form a large tetrahedron with edges of length 2. The object of the puzzle will be unchanged: arrange the pieces so that all four colors occur on each face including the bottom.

We will call this new combinatorial puzzle Cleopatra's Pyramid. Of course, for a particular coloring scheme, there may be no solution. So we will attempt to obtain a coloring scheme which is in some sense "best". The criteria for the "best" coloring scheme surely must include a minimum number of solutions.

In order to count the number of possible arrangments of the pieces, we label the octahedron and four tetrahedra A, B, C, D, E, respectively, and consider the number of possible positions for each. We begin with piece A: Four of the eight faces are visible in any admissable assembly of the pieces. No two of these four faces share an edge. Therefore, A can be placed in just two distinct positions. (We do not consider rotations which leave the same faces visible to be different arrangements.)

There are four possible faces of piece B which are to be hidden when placed adjacent to A, three rotations, and four possible faces on A which can be covered by a face of B. Hence there are $(4\cdot 3)4$ distinct choices for placing piece B. Pieces C, D, and E can be treated similarly, with the only changes being the number of available faces on A which can be covered by a face of C, D, and E, respectively. These numbers are 3, 2, and 1, so piece C produces $(4\cdot 3)3$ distinct choices, piece D, $(4\cdot 3)2$ distinct choices, and piece E, $(4\cdot 3)$ distinct choices. Combining all these options yields $2\cdot(4\cdot 3)\cdot 4\cdot(4\cdot 3)\cdot 3\cdot(4\cdot 3)\cdot 2\cdot(4\cdot 3)=4(12)^5$ different arrangements. Compared to Instant Insanity with $2(12)^4$ arrangements, we have the distinct possiblity of a more difficult game.

Since our goal is to obtain a coloring scheme which will afford a minimum number of solutions, there are several restrictions that are almost obvious:

(A) No two pieces should be identical.

(B) At least 3 colors should be on each of the tetrahedra.

(C) At least 3 colors should be on the visible faces of A in any solution.

To show these patterns more clearly we represent a color distribution by an ordered four-tuple of integers listing the number of red, yellow, blue, and green faces. So $(3,1,0,0)$ describes a tetrahedron with 3 red faces and 1 yellow face. Condition (A) prevents an interchange of identical pieces in a solution which would yield a second solution. Condition (B) prevents multiple solutions arising by applying rotations to the tetrahedra. (Note that pieces with color distributions $(4,0,0,0)$, $(3,1,0,0)$, or $(2,2,0,0)$ can be rotated with no change in the color of the visible faces.) Condition (C) prevents multiple solutions arising from rotating the octahedron. So in any solution the visible faces of A must have the distribution $(2,1,1,0)$ or $(1,1,1,1)$ up to a reordering of the colors. If three of the tetrahedra bear all four colors, then at least two of the three pieces are identical. So we can also require:

(D) At most two of the tetrahedra have all four colors.

Or, equivalently,

(D′) At least two of the tetrahedra have only three colors.

A convenient device to use in order to consider all possible coloring schemes is to write the color distribution of the *visible* faces of the tetrahedra B, C, D, E as rows of a matrix whose columns, labelled r, y, b, and g, represent the colors red, yellow, blue and green. For example, the color-distribution matrix

	r	y	b	g
B	1	2	0	0
C	1	0	2	0
D	1	0	0	2
E	1	1	1	0

denotes a coloring in which B has 1 red and 2 yellow faces visible, C shows 1 red and 2 blue faces, etc. Let $C=(c_1,c_2,c_3,c_4)$ be the column sums of such a matrix. Then $c_i \leqq 4$, lest some face of the assembled tetrahedron have a repeated color. Also according to condition (C), c_i must be $(4,3,3,2)$ or $(3,3,3,3)$ in some order, to correspond to color distribution on A of $(2,1,1,0)$ or $(1,1,1,1)$.

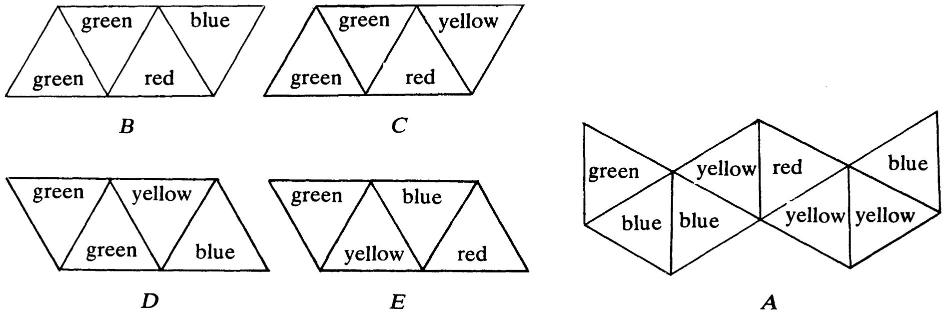
Since any permutation of rows is just a renaming of the pieces and any permutation of columns with the same column sum is just a renaming of the colors, we need only consider a relatively small number of matrices. There are 26 such matrices with column sums $(4,3,3,2)$ and only three with column sums $(3,3,3,3)$. (Two identical rows with an entry 2 are not considered since there is only one cyclic permutation of three objects, two alike.)

Any coloring scheme can now be expressed, up to permutations, by a matrix $P+Q$, where P is the color distribution matrix and Q is a 4×4 matrix representing the hidden faces of B, C, D, E, and a vector $R=((r_1,r_2,r_3,r_4),(r_5,r_6,r_7,r_8))$, where (r_1,r_2,r_3,r_4) is the color distribution for one choice of position for A and (r_5,r_6,r_7,r_8) is the color distribution for the second choice. We note that the row sums of Q must all be 1. With each of the 29 choices for P except one, we obtain at least 20 solutions to the puzzle. However, with

$$P=\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix},\quad Q=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},\quad S=P+Q=\begin{bmatrix}1&1&1&1\\1&1&0&2\\1&0&1&2\\0&1&1&2\end{bmatrix},$$

and $R=((3,3,3,3),(0,2,2,0))$ there are only 19 solutions.

Inspection of S shows there is only one way of choosing Q so that $S-Q$ has column sums $(3,3,3,3)$, and no ways of selecting Q so $S-Q$ has column sums $(4,2,2,4)$. Hence there is only one position for the octahedron A, and all faces of the tetrahedra which are not visible in a solution are green. Moreover, the coloring scheme for C, D, E is unique, and there are two distinct schemes for B. Also, the visible faces of A can be colored in two ways, the remaining faces in six ways, none of which contribute to the number of solutions of the puzzle. Hence the scheme shown in FIGURE 2 is used for



A coloring scheme for the pieces of Cleopatra's Pyramid that produces a puzzle slightly more complex than Instant Insanity. (Pieces are formed by folding with the colors out.)

FIGURE 2.

the pieces. Although this puzzle can be solved readily by trial and error, perhaps the reader might like to apply the graph techniques [2] used to solve Instant Insanity.

Serendipity occurs in a rather unique way in this topic. Several months after determining that the coloring scheme above is best, a colleague pointed out that the puzzle could be solved with the altered rule: only one color can appear on each face. In fact, the solution is unique with this rule. Furthermore, two solutions of the original puzzle can be obtained from the latter puzzle by merely rotating each small tetrahedra about the line normal to its hidden face. Either make all rotations clockwise or make all rotations counterclockwise.

Recall that there are $4(12)^5$ and $2(12)^4$ distinct arrangements for Cleopatra's Pyramid and Instant Insanity, respectively, and that there are 19 and 1 solutions respectively. Hence, although the ultimate goal of a unique solution to Cleopatra's Pyramid has eluded us, we have still constructed a puzzle 24/19 times as difficult as Instant Insanity.

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Another Strategy for SIM

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The game of SIM [1, 2, 3] is based on the fact that if the $\binom{6}{2} = 15$ lines determined by 6 points are colored with two colors, then a monochromatic triangle must occur. In SIM, two players choose a color and alternately color one of the 15 lines. The first to complete a monochromatic triangle loses the game. In this paper we present a strategy for the second player and introduce the projective geometry dual game which we call TRI-NOT.

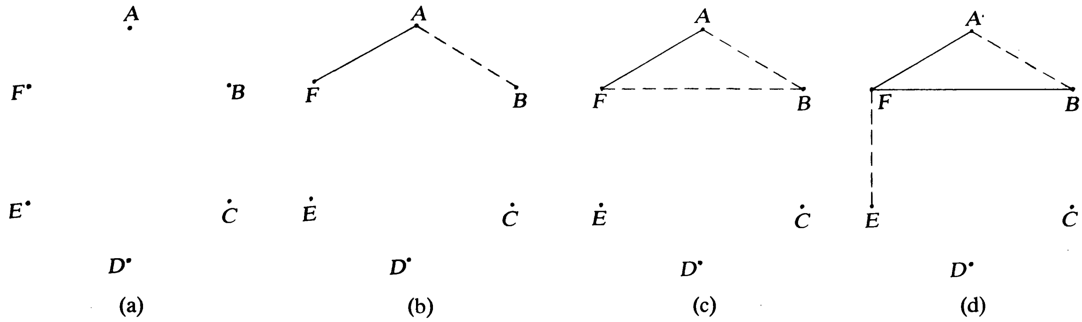


FIGURE 1.

There are several strategies for SIM in print [**4**, **5**]. However, the strategy presented here is the first easily used strategy, for it requires calculations on only three plays. For convenience lable the 6 points $A, B, \ldots, F$ as in FIGURE 1(a). Let R (red — denoted by a solid line) be the first player, B (blue — denoted by a broken line) the second player. There are four rules for B to follow. Rules 1 and 2 actually are special cases of Rule 3, but are easily stated and allow the first 2 moves by B to be made without counting.

Rule 1. On move #2, B plays any line adjacent to the line chosen by R (FIGURE 1(b)).

Rule 2. On move #4 there are two cases:

(a) if R has not played to complete a triangle, i.e., BF in FIGURE 1(b), then B plays BF (FIGURE 1(c)).

(b) if R has played BF, then B plays any line containing the vertex F, i.e., B uses the vertex with red degree 2 (FIGURE 1(d)).

On moves #6, #8, and #10 a counting scheme is employed based on the playable lines, those that do not lead to an immediate loss. Let $\mathscr{R}$ be the lines playable by R only, $\mathscr{B}$, those playable by B only, $\mathscr{R} \vee \mathscr{B}$, those playable by either R or B, and $\mathscr{N}$, those not playable by either R or B. Suppose P, Q, S are three of the six vertices and that B has already played QS. Suppose that B is ready to choose his next move. If both PQ and PS are in $(\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}$, i.e., both playable by B, we say line PS is "lost by B" upon play of PQ. A brief experience with SIM suggest that a player should "lose" as few lines as possible. With this experience in mind we define several variables which will be used in the strategy.

Let x_{ij} be the number of lines "lost by B" upon play of line ij (where $i, j \in \{A, B, C, D, E, F\}$); x_{ij} is the number of lines which would be transferred from $(\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}$ to $\mathscr{R} \cup \mathscr{N}$ if B plays line ij. Our strategy will select a line from $\mathscr{R} \vee \mathscr{B}$ with minimum x value for moves #6, #8 and #10. If there is not a unique line in $\mathscr{R} \vee \mathscr{B}$ with minimum x value, we will need a finer distinction between essentially different lines with minimum x values.

If P, Q, S are three vertices and B can play all three of PQ, QS, PS, then if B plays PQ, x_{QS} and x_{PS} are each increased by 1. If R has played QS, and both R and B can play each of PQ and PS, then if B plays PQ the number $\bar{x}_{PS}$ (the number of lines lost by R) is reduced by 1. If neither of these apply, there is no change in the x or $\bar{x}$ numbers.

Hence we define

$$y_{ijk} = \begin{cases} 2, & \text{if lines } ik \text{ and } jk \in (\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}, \\ 1, & \text{if one of } ik, jk \text{ is red, the other in } \mathscr{R} \cup (\mathscr{R} \vee \mathscr{B}), \\ 0, & \text{otherwise,} \end{cases}$$

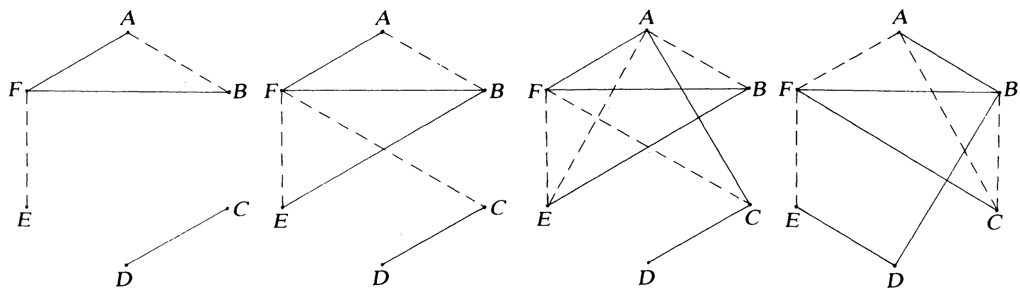
and let $z_{ij} = \sum_{k \neq i,j} y_{ijk}$. This permits us to return to the rules for the strategy.

Rule 3. On moves #6 and #8, B selects the subset of $\mathscr{R} \vee \mathscr{B}$ with minimum x_{ij} value and from this subset any line with minimum z_{ij} value.

Rule 3 is not nearly as difficult to use as it is to state precisely. B merely calculates the x value for all lines in $\mathcal{R} \vee \mathcal{B}$ and changes the entries in $\mathcal{R} \vee \mathcal{B}$ to ordered pairs such as CD, 2 where $2 = x_{CD}$. If there are several with minimum x value then the z_{ij} are calculated. This is easily done by omitting i, j and proceeding in sequence around the remaining vertices adding 0, 1 or 2 for each vertex. An example will be given below to exhibit the counting scheme.

Rule 4. If at move #10 there is one line in $\mathcal{R}$, 2 in $\mathcal{R} \vee \mathcal{B}$ and 3 in $\mathcal{B}$, then select one line from $\mathcal{R} \vee \mathcal{B}$ and two lines from $\mathcal{B}$ such that all three lines can be played without a loss. Then play the line chosen from $\mathcal{R} \vee \mathcal{B}$ on move #10, the other two lines in any order on move #12 and #14. Otherwise use Rule 3 for move #10.

At the twelfth move, player B can select a move by inspection which will gurantee a win. (It may be necessary for the inexperienced player to construct the tree of possibilities starting with move #12. This is a relatively simple task. With some practice a player can pose several "If I play —, then R plays —, and I play — and guarantee a win" statements. Remember it is only necessary on move #12 to guarantee one more move than R can make!)



$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
	AC 1,4	
	AD	
	AE 1,4	
	BC	
	BD	
	BE	
	CE 1,5	
	CF 1,3	
	DE	
	DF	

(a)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
CE	*AC*, 1	
	AD, 1	
	AE, 0	
	BC, 1	
	BD, 1	
	DE, 1	
	DF, 1	

(b)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
CE	*BC*	*AD*
	BD	
	DE	
	DF	

(c)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
AE	*CD*, 1,1	*AD*
	CE, 1,1	*BE*
		DF

(d)

FIGURE 2.

An example should help clarify these rules. Suppose the sequence of moves has been Red *AF*, Blue *AB*, Red *BF*, Blue *FE*, Red *CD*, so that the state of the game is as given in FIGURE 2(a). Note *A* and *B* are equivalent points as are *C* and *D*. So we need only consider *AC*, *AE*, *CE*, *CF*, each of which has x value 1. In calculating the z values, for *AC* we add 0 for *B*, 1 for *D*, 2 for *E* and 1 for *F*. Hence, $z_{AC} = 4$. For *CF*, we count 1 for *A*, 1 for *B*, 1 for *D* and 0 for *E*. Hence, $z_{CF} = 3$ and *CF* is the best move. Suppose B plays *CF* and R plays *BE*. The state of the game and the revised table will then be as in FIGURE 2(b).

There is no need to calculate the z values since $x_{AE} = 0$ is the unique minimum x value. So B plays *AE*. Suppose *R* plays *AC*. The state of the game and the revised table are then as shown in FIGURE 2(c). Actually, R has played quite well, forcing B to be very careful. Since the special case for move #10 governed by Rule 4 does not apply, B uses Rule 3 again. This requires B to play *BC*. Consider *DF* and AD played in that order by B. B can play both lines without losing. The only way that R can block this winning sequence is by playing DF on move #11. But in this case R loses BD and B plays DE on move #12 and BD on move #14. Hence B wins the game.

Although in this example Rule 4 degenerated to Rule 3, the improper use of Rule 3 on the 10th move can be disastrous. For example, consider the game situation given in FIGURE 2(d). Rule 3 would indicate that either choice of CD or CE would be acceptable for B. However, if B plays CE, Red can play CD and AE while B can play only one of AD and DF. Hence R wins!

Testing our strategy seems to require looking at the entire game tree. The only simplification possible is to reduce the number of cases to a minimum. There are several ways to "prune" the game tree to reasonable size. The first method is to prune branches that are topologically (graph theoretically) isomorphic. In SIM this method reduces the number of terminal vertices from DeLoach's 1.7×10^{10} **[6]** to Beeler's 2250 **[7]**. The second method of pruning a tree is to eliminate branches that are not permitted by the strategy. The third method of pruning trees is to eliminate all but one of several isomorphic states since we are interested only in the present state of the game and not in its history. Finally, those branches which result in a premature loss by R may also be pruned.

The first and third methods are relatively easy to check by hand. The second pruning method requires some calculations in the case of SIM. These calculations have been made by hand and by computer as a check, producing the following table of nonisomorphic moves after each move through move 9:

move number	1	2	3	4	5	6	7	8	9
nonisomorphic games	1	1	5	4	15	13	56	41	97

(The second pruning method is largely responsible for keeping the number of games associated with the "even move numbers" in this table no larger than for the preceding odd number.)

Most of the 97 cases after move #9 can be dealt with by direct reasoning. In fact there are only 19 cases in which R can block a winning move by B, forcing B to choose an alternate winning move. The process of checking isomorphisms and applying the strategy is too long to present here. However, the process has been done by hand several times, requiring about four hours each time.

The strategy suggested in this note was originally obtained and tested using the projective geometry dual of SIM, which has been named TRI-NOT by the author's children. Instead of 6 points, no 3 collinear, TRI-NOT is defined by 6 lines, no 3 concurrent, and the $\binom{6}{2} = 15$ points of intersection. The dual of a triangle is a triangle, but instead of edges, it is the vertices that are colored. The first to color the vertices of a triangle loses the game. The advantage of TRI-NOT is that it can be made into a board game with reusable pegs or markers (FIGURE 3). Our strategy translates directly to a strategy for TRI-NOT via the duality principle.

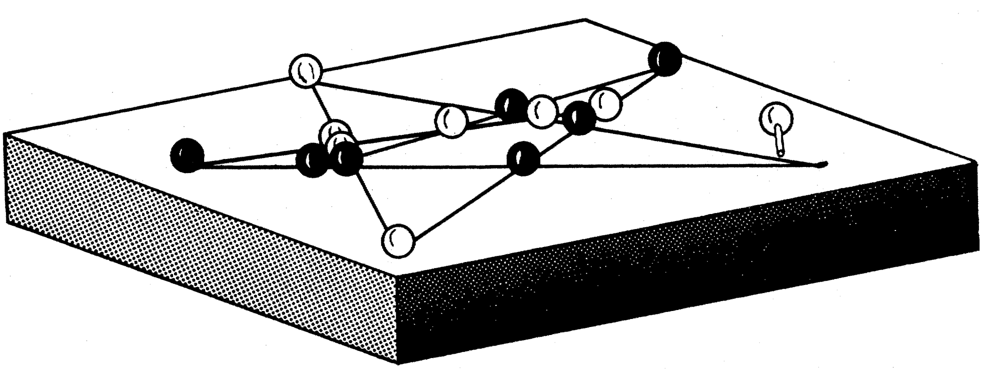


FIGURE 3.

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Square Permutations

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This paper presents a new idea called square permutations, originally suggested by David Silverman. The development is elementary. It can be fully understood by a college sophomore, most of it even by a bright high school student. This illustrates my claim **[2, 3]** that original mathematical research can take place at a much lower level than is commonly realized.

Let P denote a permutation of the integers $0, 1, 2, \ldots, n$. We shall write permutations in cycle notation. For example, the permutation

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P(k)$	4	8	14	6	0	11	3	9	1	7	15	5	13	12	2	10

will be written (0 4)(1 8)(2 14)(3 6)(5 11)(7 9)(10 15)(12 13). We shall say that P is a **square permutation** if and only if $k + P(k)$ is a perfect square for each value of k from 0 up to n. We shall say that n has a square permutation if and only if there is such a P. The permutation above is an example of a square permutation; it shows that 15 has a square permutation. For our main result we will prove below that every non-negative integer has a square permutation.

LEMMA 1. *For $n = 1, 2, \ldots,$ both $n^2 - 1$ and n^2 have square permutations.*

Proof. The permutations are, respectively, $(0)\ (1 \quad n^2-1)(2 \quad n^2-2)\ldots$ and $(0 \quad n^2)(1 \quad n^2-1)$ $(2 \quad n^2-2)\ldots$, where the last cycle is $(\frac{1}{2}n^2)$ if n is even, and $(\frac{1}{2}(n^2-1) \quad \frac{1}{2}(n^2+1))$ if n is odd.

From small square permutations we can sometimes build up larger ones by appending cycles of length two whose elements sum to a square. For example, from (0 4) (1 3) (2) (the permutation for 4 obtained from Lemma 1) we can get a square permutation for 11 by appending (5 11) (6 10) (7 9) (8). This method generalizes:

LEMMA 2. *Suppose that m has a square permutation, that $n > m \geqq 0$, and that for some integer r, $n + m = r^2 - 1$. Then n has a square permutation.*

Proof. Append to the square permutation for m the cycles $(m+1 \quad n)(m+2 \quad n-1)\ldots$; the last cycle is $(\frac{1}{2}r^2)$ or $(\frac{1}{2}(r^2-1) \quad \frac{1}{2}(r^2+1))$ according as r is even or odd.

THEOREM. *Every non-negative integer has a square permutation.*

The proof proceeds by mathematical induction. From Lemma 1 we know that 0, 1, 3, and 4 have square permutations. Obviously (0 1) (2) is a square permutation for 2. Let $n \geqq 5$, and assume that the integers $0, 1, 2, \ldots, n-1$ all have square permutations. From Lemma 1, if n is a square or one less than a square, it has a square permutation. Therefore suppose that $(r-1)^2 < n < r^2 - 1$ for some integer $r > 2$. We shall show that the integer $m = r^2 - 1 - n$ meets the conditions of Lemma 2. By the supposition $n < r^2 - 1$, we have $m > 0$; and by the induction assumption, if $m < n$, then m has a square permutation. It remains to show that $m < n$. We prove this as follows. From $(r-2)^2 \geqq 1$ we get $2r - 2 \leqq (r-1)^2$. Then

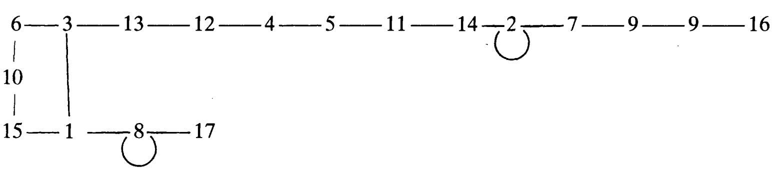
$$m < r^2 - 1 - (r-1)^2 = 2r - 2 \leqq (r-1)^2 < n.$$

So, from Lemma 2, n has a square permutation. The theorem follows.

Variations immediately suggest themselves. For example, consider permutations of the set $1, 2, \ldots, n$ (zero omitted) that satisfy the square condition on $k + P(k)$ for all k. Call such a permutation a **positive square permutation**. Do positive square permutations exist for all values of n? It turns out that there are no positive square permutations for $n = 1, 2, 4, 6, 7$, or 11, but there are for all other values of n. We briefly sketch the argument. The impossibility for $n = 1, 2$, or 4 is easy to see. For $n = 6$ or 7 we must have $P(1) = 3 = P(6)$, a contradiction. Finally, for $n = 11$, the contradiction is $P(4) = 5 = P(11)$. There is a bit more to showing that all other values of n have positive square permutations. Lemma 2 remains valid for positive square permutations, and its proof does not require any change. Lemma 1 must, however, be modified; we leave its proof to the reader:

LEMMA 3. *For $n = 2, 3, \ldots$, the integer $n^2 - 1$ has a positive square permutation.*

The lemmas are still not adequate to answer completely the question of what values of n have positive square permutations. One other device that can be used is the following. Construct a graph whose vertices are the integers $1, 2, \ldots, n$, and which has an edge joining vertices i and j if and only if $i + j$ is a perfect square. Then by inspection, decompose the graph into disjoint cycles (if possible). This will yield a positive square permutation. For example, the graph below



gives the following positive square permutation for 17:

$$(1\ 3\ 6\ 10\ 15)(4\ 5)(8\ 17)(9\ 16)(11\ 14)(12\ 13)(2\ 7).$$

The graphical procedure is practical for values of n up to about 24. Beyond that, the graphs become too complicated to analyse visually.

By using Lemmas 2 and 3 and the graphical procedure, the reader should now be able to find positive square permutations for all integers up to 31, except those listed above. For larger values an inductive proof can be constructed.

Lemmas 1, 2, and 3 produce primarily square permutations whose cycles have length at most two. Graphical methods can be used to find square permutations with longer cycles. For example, for $n = 20$, a positive square permutation is

$$(1\ 8\ 17\ 19\ 6\ 10\ 15)(2\ 7\ 9\ 16\ 20\ 5\ 11\ 14)(3\ 13)(4\ 12)(18),$$

while for $n = 17$ a square permutation with a cycle of length 16 is

$$(0\ 1\ 15\ 10\ 6\ 3\ 13\ 12\ 4\ 5\ 11\ 14\ 2\ 7\ 9\ 16)(8\ 17).$$

We conjecture that arbitrarily long cycles exist in square permutations. In partial confirmation Michael Razar has proved (but not published) that arbitrarily long cycles exist for which $k + P(k)$ is square. The proof uses advanced methods.

What has been done with squares can be done with primes, or indeed with any subset of the non-negative integers. We say that a permutation P is a (positive) **prime permutation for** $\boldsymbol{n}$ if and only if it is a permutation of the integers $1,2,\dots,n$ such that $k+P(k)$ is a prime for all $k=1,2,\dots,n$. Then we can prove that every positive integer has a positive prime permutation. The proof requires a simple modification of Lemma 2, plus Bertrand's Theorem [**1**] that there is a prime between n and $2n$ (inclusive) for all non-negative n. In the same way, it also follows that one can always construct a prime permutation all of whose cycles have order 2. On the other hand, there are prime permutations with long cycles. For $n=18$, the following positive prime permutation of a single cycle was constructed using the graphical method:

$$(1\ 6\ 5\ 8\ 3\ 2\ 17\ 12\ 11\ 18\ 13\ 10\ 7\ 16\ 15\ 14\ 9\ 4).$$

The author thanks the referee for several helpful suggestions, in particular, to distinguish between square permutations and positive square permutations.

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Tetrahedral Models from Envelopes

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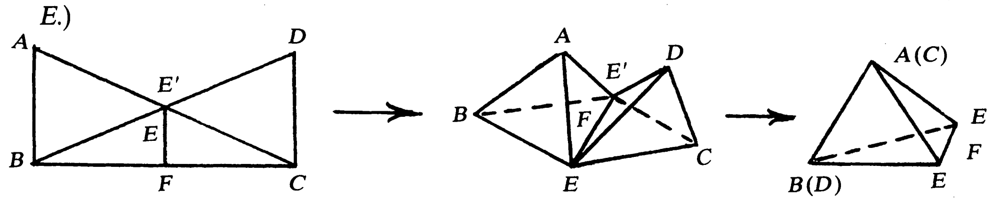
Essentially, a sealed mailing envelope consists of two superposed congruent rectangles joined at their edges. Tetrahedral models have been formed from envelopes in a variety of ways [**1, 2, 3, 4, 5**], but the following procedure is neater; see FIGURE 1.

1. To facilitate folding, score the diagonals of a sealed envelope with a dry ball-point pen or a scissors blade, taking care to compress the fibers of the paper without tearing them.
2. Cut along two half-diagonals to remove the sector containing a long side and the envelope flap.
3. Fold over the remaining portion along the half-diagonals and crease firmly. Fold back along the same lines and crease firmly again. (Avoid envelopes on which the diagonals fall along a sealed seam.)
4. Bring DC onto AB, flatten the envelope to form EF and crease firmly. Fold back along EF and again crease firmly. (In the figure, E' indicates the point on the lower rectangle directly under E.)



From envelope to tetrahedron
FIGURE 1.

5. Separate E and E' until EFE' is a straight line. Fold up around EFE' until D meets A, thus forming a hexahedron.
6. Tuck D under AB (or A under DC) and press up on B and C until DC and BA coincide.

The rigid, self-stabilized model with no open edges thus produced can be collapsed for storage or transportation and reconstituted when desired without staples or adhesives. The tetrahedron is a bisphenoid or double wedge, a special isosceles tetrahedron.

The envelope must have $r = BC/AB > 1$. The larger r is, the more elongated the tetrahedron. The smaller r is, the flatter the tetrahedron. When $r = 1$, the cut envelope flattens into three superposed squares. For the average small business envelope, r is about 1.76 which deviates little from $\sqrt{3}$, so the tetrahedron formed from it is almost regular.

Raw material for the model making is plentiful in used envelopes slit open along the long flap edge, and in the return envelopes so often included in junk mail. The ratio r can always be modified by cutting the envelope. If an envelope of the desired size and r is not available, one can always be made by folding a rectangular sheet of the proper dimensions into two superposed rectangles and joining them along the shorter edges with sticky tape or, preferably, with flaps previously provided for gluing. The larger the envelope, the stiffer the paper should be.

When told about this construction method, Mary Krimmel of La Jolla, California, was so enthusiastic that she used new envelopes to make tetrahedral place cards for her next dinner party. Messages were placed inside in Chinese fortune cookie fashion.

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A Child's Game with Permutations

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War is a child's card game played with a poker deck. After the deck is shuffled and dealt to the two players, they begin matching cards with the higher card capturing the lower card. Stalemates are broken according to local rules and the game is over when one player has all of the cards. During a recent game my seven-year-old son asked, "Does this game ever have to end?" Since we could not answer his question within the context of war (the number of cards and stalemates were troublesome) we changed the rules a bit and considered the following game.

Let n be an even positive integer and let

$$\pi = \begin{pmatrix} 1 , & 2 , & \ldots, & n \\ \pi(1), & \pi(2), & \ldots, & \pi(n) \end{pmatrix}$$

be an element of S_n, the set of permutations of n symbols. We deal π into two initial hands, $H_1^{(0)} = (\pi(1), \pi(3), \ldots, \pi(n-1))$ and $H_2^{(0)} = (\pi(2), \pi(4), \ldots, \pi(n))$, and proceed according to the rule: The leading integers of each hand are placed, in decreasing order, at the end of the hand containing the larger of the two. For example, if $\pi(1) > \pi(2)$, then after one round we have $H_1^{(1)} =$

$(\pi(3), \ldots, \pi(n-1), \pi(1), \pi(2))$ and $H_2^{(1)} = (\pi(4), \ldots, \pi(n))$. An element of S_n will be referred to as a **game.** A game is said to **terminate** if one of the hands is empty after a finite number of matches.

Must every game in S_n terminate? For S_2, S_4 and S_6 the answer is easily seen to be yes. However, by taking $n = 10$ and

$$\pi = \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10 \\ 8, & 3, & 10, & 5, & 7, & 9, & 2, & 6, & 4, & 1 \end{pmatrix}$$

one can see that the answer in general is no, since $H_1^{(4)} = H_1^{(64)} = (4,8,3,10,5)$ and $H_2^{(4)} = H_2^{(64)} = (1,9,7,6,2)$. (Use a computer if you find the game unexciting.) In fact, a game fails to terminate if and only if there exist positive integers r and s such that $H_1^{(r)} = H_1^{(s)}$ and $H_2^{(r)} = H_2^{(s)}$. In this case we say the game is **cyclic** and define its **length** to be $\min_{r<s} \{s - r \mid H_1^{(r)} = H_1^{(s)} \text{ and } H_2^{(r)} = H_2^{(s)}\}$. The length of the game π is 60. If one is a group theorist at heart, then one quickly observes that the index 10 of S_{10} divides (integrally) the length of π.

Another surprising bit of regularity exhibited by π can be seen in the following array:

$$\begin{aligned} H_1^{(4)} &= (4,8,3,10,5), & H_2^{(4)} &= (1,9,7,6,2) \\ H_1^{(24)} &= (4,8,2,10,5), & H_2^{(24)} &= (3,9,7,6,1) \\ H_1^{(44)} &= (4,8,1,10,5), & H_2^{(44)} &= (2,9,7,6,3) \\ H_1^{(64)} &= (4,8,3,10,5), & H_2^{(64)} &= (1,9,7,6,2). \end{aligned}$$

The state of the game after match $m + 20$ (for $m \geqq 4$) can be determined from the state of the game after match m by replacing 1 with 3, 2 with 1 and 3 with 2; in other words, the permutation $(1,3,2)$ is associated with the game π in a natural way. For lack of a better term, we refer to 20 as the **sublength** of π. The sublength of π is divisible by the index of S_{10} and the length of π is the product of its sublength and the group-theoretic order of the associated permutation.

The cyclic game π was obtained by having a computer generate and play a random sample of 41 games chosen from each of $S_8, S_{10}, S_{12}, S_{14}$ and S_{16}. (A sample of size 41 was chosen because it provides, without requiring a great deal of computer time, an 80% confidence level that the actual probability of termination does not vary from the sample probability by more than .1.) Three games (including π) of the sample from S_{10} were cyclic; each had length 60, sublength 20 and associated permutation $(1,2,3)$. The sample from S_{14} contained two cyclic games of length 112 and sublength 28. The associated permutations were different in this case: $(1,3,2,4)(12,14)$ and $(1,4,3,2)(12,14)$. Each game of the samples from S_8, S_{12} and S_{16} terminated in at most 24, 50 and 112 matches, respectively.

Here are some conjectures and problems for the reader (or his computer) to consider:

1. If $n \geqq 10$ and not divisible by four, then there exists a game in S_n which is cyclic. Each cyclic game in S_n has length $l = 2nm$ where $2n$ is the sublength of the game, m is the group-theoretic order of the associated permutation and $(n+2)/4 \leqq m \leqq n$.
2. If $n \geqq 12$ and divisible by four, then each game in S_n terminates. Is there a non-trivial upper bound for the number of matches required to guarantee termination?
3. Finally, as n tends to infinity, does the probability of a game terminating tend to one?

Of course, the basic problem is to establish a useful means (short of playing the game) for deciding whether a given game terminates or is cyclic. While the author and his students have been unsuccessful in solving this problem, there does seem to be some common denominator for the cyclic games. After examining the cyclic games in the samples from S_{10} and S_{14} we were able to construct the cyclic game

$$\pi = \begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18 \\ 6,4,16,13,7,3,8,17,1,12,18,14,11,2,10,15,5,9 \end{pmatrix}$$

with minimum difficulty. Sure enough, this game, an element of S_{18}, has length 360, sublength 36 and associated permutation $(1,2,3,4,5)(16,18)(15,17)$.

PROBLEMS

DAN EUSTICE, Editor
LEROY F. MEYERS, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before October 1, 1978

1029.* Does there exist any prime number such that if any digit (in base 10) is changed to any other digit, the resulting number is always composite? [*Murray S. Klamkin, University of Alberta.*]

1030. a. Solve the following functional-differential equation for the complex-valued differentiable function f:

$$f(s+t)=f(s)+f(t)-f'(s)f'(t) \text{ for all real } s \text{ and } t,$$

and $f(0)=0$.

b. If the real part of $f(t)$ is non-positive for all real t, but f is not identically zero, show that $f(t)=0$ only if $t=0$.

(This problem arose in connection with "infinitely divisible distributions" in probability.) [*G. Edgar, The Ohio State University.*]

1031. There are n people, numbered consecutively, standing in a circle. First #2 sits down, then #4, #6, etc., continuing around the circle with every other standing person sitting down until just one person is left standing. What is his number ? (For example, with $n=6$, the seating order is 2, 4, 6, 3, 1 and 5 is left standing.) [*Richard A. Gibbs, Fort Lewis College.*]

1032. Let $l_1(x)=\log x$, $l_2(x)=\log\log x$, and $l_k(x)=\log l_{k-1}(x)$. Let $N(k)$ be the first integer n such that $l_k(n)>1$. When k is fixed, the integral test shows that the series

$$(\#) \qquad \sum_{n=N(k)}^{\infty} \frac{1}{n l_1(n) l_2(n)\cdots(l_k(n))^p}$$

diverges for $p=1$ and converges for $p>1$. R. P. Agnew [Amer. Math. Monthly, 54 (1947), 273–274; Selected papers on calculus, MAA 1969, pp. 348–349] called attention to the result that (#) is very slowly divergent if $p=1$ and k (the number of logarithmic factors in (#)) is no longer fixed but depends on n, being taken as large as possible so that all the logarithms exceed 1, i.e., so that $l_k(n)>1$ but $l_{k+1}(n)<1$. With this choice of $k=k(n)$, how large can $p=p(k)$ be before the series becomes convergent? (Will $p=2$ or $p=k$ suffice?) [*R. P. Boas, Northwestern University.*]

ASSISTANT EDITORS: DON BONAR, *Denison University*: WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

Solutions

Different Number of Prime Divisors **May 1976**

983*. Are there arbitrarily long sequences of consecutive integers no two of which have the same number of prime divisors? [*Bernardo Recamán, Bogotá, Colombia.*]

Solution: Denote by $d(n)$ the number of divisors of n. L. Mirsky and I proved (*The distribution of values of the divisor function d(n)*, Proc. London Math. Soc. Ser. III, 2 (1952), 257–271) that for infinitely many n the values $d(n), d(n+1), \ldots, d(n+c(\log n)/\log\log n)$ are all different. If the same method is applied to $v(n)$, the number of distinct prime factors of n, we obtain that there are infinitely many n's for which all of the $v(n+i)$, $1 \leqq i < (c(\log n)/\log\log n)^{1/2}$, are different. This answers the problem affirmatively.

Very likely for infinitely many n's all the numbers $v(n+i)$, $0 \leqq i < c_1(\log n)/(\log\log n)^{c_2}$, are all different, but I cannot prove this. This result cannot hold for $c_2 < 1$, for it follows from the prime number theorem that

$$\max_{1 \leqq m \leqq n} v(m) < (1+o(1))\frac{\log n}{\log\log n}.$$

Finally, it follows from the well-known result of Kac and myself (Kubilius, *Probabilistic number theory*, Amer. Math. Soc. translation) that for every k, the density of integers n for which the numbers $v(n+i)$, $1 \leqq i \leqq k$, are all distinct is 1. The same result holds if $k = k_n$. $k_n = o((\log\log n)^{1/2})$.

PAUL ERDÖS
Hungarian Academy of Science

Transcendental or Rational **May 1976**

985. Let $Q_k = 1/(k+2)! + 2/(k+3)! + 3/(k+4)! + \cdots$. Show that Q_k is transcendental for all positive integers k, but rational for $k = 0$. [*Jeffrey Shallit, Princeton University.*]

Solution: Let $g(k,x) = \Sigma_{n=1}^{\infty} nx^{k+n+1}/(k+n+1)!$. Then $g'(k,x) = g(k-1,x)$, $g(-1,x) = xe^x$ and $g(k,0) = 0$ if $k > -2$. Thus,

$$g(0,x) = \int_0^x ue^u du = (x-1)e^x + 1$$

and inductively we find $g(k,x) = (x-k-1)e^x + \Sigma_{j=0}^{k}(k-j+1)x^j/j!$. Therefore, $Q_k = g(k,1) = -ke + $(rational number). This means that Q_k is rational if $k = 0$ and transcendental otherwise.

DANNY GOLDSTEIN, grade eight
Heritage Junior High School
Livingston, New Jersey

Also solved by J. C. Binz (Switzerland), Erhard Braune (Austria), A. Deutsch, Ragnar Dybvik (Norway), Marguerite F. Gerstell, M. G. Greening (Australia), Richard A. Groeneveld, G. A. Heuer, Hans Kappus (Germany), Eli Leon Isaacson, Jordan I. Levy, Lael F. Kinch, Lew Kowarski, N. J. Kuenzi & Bob Prielipp, Graham Lord (Canada), Jerry M. Metzger, V. Srinivas (India), J. M. Stark, James W. Walker, Edward T. H. Wang (Canada), and the proposer.

A Mean Value **May 1976**

987. Let f be differentiable with f' continuous on $[a, b]$. Show that if there is a number c in $(a, b]$ such that $f'(c) = 0$, then we can find a number ξ in (a, b) such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{b - a}$$

[*Sidney Penner, Bronx Community College.*]

Solution: For x in $[a, b]$, define

$$g(x) = f'(x) - \left[\frac{f(x) - f(a)}{b - a}\right].$$

Assume first that $f(c) > f(a)$. Choose d in $(a, c]$ such that $f(d)$ is a maximum for f in $[a, c]$, and let e be a point in (a, d) such that $f'(e) = (f(d) - f(a))/(d - a)$. Then $g(e) > 0 > g(d)$ so, since g is continuous, there is a point ξ in (e, d) such that $g(\xi) = 0$. A similar argument takes care of the cases $f(c) < f(a)$ and $f(c) = f(a)$.

St. Olaf Problem Group
St. Olaf College

Also solved by R. P. Boas, Dale Brenneman, Daniel S. Freed, Donald C. Fuller, Landy Godbold, Richard A. Groeneveld, G. A. Heuer, C. H. Heiberg, W. I. Humkevan, Eli Leon Isaacson, Richard Johnsonbaugh, Virgil C. Kowalik, Jordan I. Levy, Adam Riese, Henry J. Schultz, Harry T. Sedinger, Joseph Silverman, J. M. Stark, Robert M. Tardiff, and the proposer. R. P. Boas points out that the hypothesis that f' is continuous is unnecessary since the derivative has the intermediate value property.

Orthogonal Projection **September 1976**

988. A given equilateral triangle ABC is projected orthogonally from a given plane P to another plane P'. Show that the sum of the squares of the sides of triangle $A'B'C'$ is independent of the orientation of the triangle ABC in plane P. [*Murray S. Klamkin, University of Alberta.*]

Solution: We associate complex variables $z = x + iy$ and $z' = x' + iy'$ with P and P' respectively. Let Π be a closed n-sided polygon in P with vertices $z_1, z_2, \ldots, z_n, z_{n+1}$ ($z_{n+1} = z_1$), and let Π' be the image of Π in P' under an affine transformation $x' = ax + by + c$, $y' = dx + ey + f$. In terms of z' and z the transformation can be written $z' = \alpha z + \beta \bar{z} + \gamma$ where α, β, γ are complex constants and $\bar{z}$ is the conjugate of z. A side $\Delta_j = z_{j+1} - z_j$ of Π will transform: $\Delta_j' = \alpha \Delta_j + \beta \bar{\Delta}_j$. Applying the "cosine law" identity

$$|u + v|^2 = |u|^2 + |v|^2 + 2\,\mathrm{Re}(u\bar{v}),$$

one obtains

$$|\Delta_j'|^2 = (|\alpha|^2 + |\beta|^2)|\Delta_j|^2 + 2\,\mathrm{Re}(\alpha\bar{\beta}\Delta_j^2)$$

for the squared length of Δ_j'. Let

$$S = \sum_{j=1}^{n} |\Delta_j|^2, \qquad S' = \sum_{j=1}^{n} |\Delta_j'|^2, \qquad \text{and} \qquad \sigma_k = \sum_{j=1}^{n} \Delta_j^k \qquad (k = 1, 2).$$

Then $S' = (|\alpha|^2 + |\beta|^2)S + 2\,\mathrm{Re}(\alpha\bar{\beta}\sigma_2)$. Rotation of Π through an angle θ in plane P will leave S unchanged while causing each Δ_j^2, and hence σ_2, to rotate through the angle 2θ. The sum S' will be fixed for all θ iff the same is true of $\mathrm{Re}(\alpha\bar{\beta}\sigma_2)$, in other words iff $\alpha\bar{\beta}\sigma_2 = 0$. We have the following theorem:

Under any affine transformation, other than a similarity transformation, for S' to be independent of the orientation of Π in P it is necessary and sufficient that

$\Delta_1, \Delta_2, \ldots, \Delta_n$ *be roots of* $\Delta^n - p_{n-3}(\Delta) = 0$, *where* p_{n-3} *is a polynomial of degree* $n - 3$ *at most.*

It is generally true that $(\Delta - \Delta_1)(\Delta - \Delta_2) \cdots (\Delta - \Delta_n) = \Delta^n - \sigma_1 \Delta^{n-1} + \frac{1}{2}(\sigma_1^2 - \sigma_2)\Delta^{n-2} - p_{n-3}(\Delta)$. In the present case, $\sigma_1 = 0$ because Π is closed; by excluding similarity (i.e., conformal) transformations we deny the value 0 to $\alpha\bar{\beta}$ so that $\alpha\bar{\beta}\sigma_2 = 0$ only if $\sigma_2 = 0$. Thus the theorem.

When $n = 3$, p_{n-3} must be a constant. This establishes the equilateral triangle as the only triangle with the O-I property (meaning that S' is orientation-independent). By reason of the cyclotomic equation, $\Delta^n = 1$, all regular polygons have the O-I property; but a polygon of more than three sides need not be regular to have it. Indeed, by the addition of a single vertex, any n-gon lacking the property can be expanded into an $(n+1)$-gon possessing it. We should mention, finally, that orthogonal projection from P to P' is a special case of affine transformation, non-conformal if the planes are not parallel.

W. WESTON MEYER
General Motors Research Laboratories

Also solved by Mangho Ahuja, Jonathan Choate, Charles Chouteau, Ragnar Dybvik (Norway), Thomas E. Elsner, Howard Eves, Daniel S. Freed, Leon Gerber, Michael Goldberg, Leonard D. Goldstone, M. G. Greening (Australia), Robert M. Hashway, Eli L. Isaacson, Vladimir F. Ivanoff, Hans Kappus (Germany), J. Pfaendtner (Germany), Harry D. Ruderman, V. Srinivas (India), St. Olaf Problem Group, Bernal P. Victor (Venezuela), Pambuccian Victor (Romania), and the proposer. There was one unsigned solution.

A Stirling Expression **September 1976**

989. Let $r \geqq 0$, $s \geqq 0$, and $r + s \leqq n$. Find the number of sequences of positive integers $(a_1, a_2, \ldots, a_n)$ such that for $1 \leqq k \leqq n$, $a_k \leqq k$ where $a_k = 1$ for r values of k, and $a_k = k$ for s values of k. [*L. Carlitz and Richard Scoville, Duke University.*]

Solution: Let $C(n, r, s)$ denote the desired number of sequences. (The restriction that $r + s \leqq n$ will be shown unnecessary.) It follows from the definition that

$$(*) \qquad C(n, r, s) = C(n-1, r-1, s) + C(n-1, r, s-1) + (n-2)C(n-1, r, s),$$

as is clear by considering the possible values for a_n. The recurrence $(*)$ holds for $n > 1$, $r \geqq 1$, and $s \geqq 1$. We also have $C(n, r, 0) = C(n, 0, s) = C(n, 0, 0) = 0$, as well as $C(n, r, s) = 0$ if $r + s > n + 1$. Let

$$A_n(x, y) = \sum_{r,s} C(n, r, s) x^r y^s.$$

Then for $n > 1$, we can substitute the right-hand side of $(*)$ in the summation, where we take $C(n-1, -1, s) = C(n-1, r, -1) = 0$. We find then that $A_n(x, y) = (x + y + n - 2)A_{n-1}(x, y)$. This with $A_1(x, y) = xy$ and $A_2(x, y) = xy(x + y)$ shows that

$$A_n(x, y) = xy(x + y)(x + y + 1) \cdots (x + y + n - 2).$$

Hence, with $x(x+1) \cdots (x + k - 1) = \sum_{j=0}^{k} S_1(k, j) x^j$, where $S_1(k, j)$ is a Stirling number of the first kind, it follows that

$$A_n(x, y) = xy \sum_{j=0}^{n-1} S_1(n-1, j)(x + y)^j = xy \sum_{r+s \leqq n-1} S_1(n-1, r+s) \binom{r+s}{r} x^r y^s.$$

Thus, we find that $C(n, r, s) = \binom{r+s-2}{r-1} S_1(n-1, r+s-2)$.

L. CARLITZ AND RICHARD SCOVILLE
Duke University

Also solved by Eli L. Isaacson and Gillian W. Valk. M. T. Bird showed that, for a given n, the total number of sequences summed over values of r and s with $r + s \leqq n$ *is* $n! - 2^{n-1}$.

REVIEWS

PAUL J. CAMPBELL, Editor
Beloit College

PIERRE MALRAISON, Editor
Control Data Corp.

Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Some reviews of books are adapted from the Telegraphic Reviews in the American Mathematical Monthly.

Iyanaga, S. and Kawada, Y., Encyclopedic Dictionary of Mathematics, MIT Pr, 1977.

Translation of the Japanese encyclopedia *Iwanami Sugaku Ziten*. Contains 435 extensively referenced and cross-referenced moderate length articles on topics similar to the major subheadings of the AMS subject classification arranged alphabetically. Half of the second volume is appendices and tables: formulas, numbers, lists of articles, name and subject indices. An invaluable up-to-date sophisticated reference it is without precedent or peer in the mathematical literature.

Dorling, A.R. (Ed.), Uses of Mathematical Literature, Butterworths, 1977.

A concise yet rich resource for anyone concerned about mathematics library collections, and much information for someone seeking guidance outside her own fields of expertise. Three chapters on general reference sources followed by 14 topical bibliographical essays by specialists. Covers several thousand mathematical references many of which are probably unfamiliar to the average mathematician.

May, Kenneth O., Index of The American Mathematical Monthly, Volumes 1 through 80 (1894-1973), MAA, 1977; vi + 269 pp, $16 (MAA member: $10).

Chronological table of contents, author index, and subject index. Not included: book and film reviews, abstracts, official announcements, annual reports (e.g., on Putnam competition), research and problem proposals, and solutions. Since the *Index* is not automatically sent to subscribers, mathematicians will have to make a special point of ordering it for their libraries.

Zeeman, E.C., Catastrophe Theory: Selected Papers 1972-1977, A-W, 1977; x + 675 pp, $26.50; $14.50 (P).

A well-edited compendium of Zeeman's papers on catastrophe theory. Papers are grouped by subject (biological, social, physical sciences, mathematics); some have been retyped, in a pleasing typeface; each section is preceded by comments; and there is an index. Only the too-wide lower margins are to be lamented. Included is a bibliography of catastrophe theory to 1974 (before the subject took off) and one new piece, "Afterthought," an imaginary dialogue between Zeeman and a critical reader.

Gorman, James, *The shape of change*, The Sciences 16 (September/October 1976) 17-22.

Popular exposition of catastrophe theory.

Zahler, Raphael S. and Sussman, Hector J., *Claims and accomplishments of applied catastrophe theory,* Nature 269 (27 October 1977) 759-763.

Thorough-going, unremitting, and specific attack on attempts to apply catastrophe theory. The list of charges: confusion about continuity, use of Thom's theorem to justify extrapolation, prediction contrary to fact, lack of true testable predictions, misuse of genericity, misleading use of mathematics, careless discussion of evidence, unreasonable or ambiguous hypotheses, spurious quantification, and better alternatives.

Kuyk, W., Complementarity in Mathematics, Reidel, 1977; 186 pp, $15.95.

A three-part inquiry into the philosophy of mathematics based on a series of lectures to mixed groups of mathematics and philosophy students. The first section treats the Gödel completeness and incompleteness theorems. The second section is a detailed historical essay on the epistemology of mathematics from ancient times through Bourbaki. The final section makes a case for a "principle of complementarity" (discrete vs. continuous, algebra vs. geometry) as a basis for a philosophy of mathematics.

Wade, Nicholas, *Scandal in the heavens: renowned astronomer accused of fraud,* Science 198 (18 November 1977) 707-708.

Robert R. Newton (Hopkins) has recently published *The Crime of Claudius Ptolemy* (Johns Hopkins U. Pr., 1977) accusing Ptolemy of faking data to support geocentric astronomical theory in the *Almagest.* Newton's charges, based on an imposing array of evidence, have nevertheless been disputed by Owen Gingerich (Harvard), who suggests Ptolemy simply selected from a larger body of data just those observations best according with his theory.

Adler, Irving, *The consequences of contact pressure in phyllotaxis,* J. Theoretical Biology 65 (1977) 29-77.

This paper comes close to an ultimate explanation of normal phyllotaxis, i.e., why the numbers of conspicuous left spirals and right spirals in a leaf distribution on a plant are usually consecutive Fibonacci numbers. Contact pressure is pressure by leaves already present to force a new leaf to develop as far away as possible. Adler offers a simple necessary and sufficient condition for contact pressure to produce normal phyllotaxis.

Wiegand, Sylvia, *Grace Chisholm Young,* Association for Women in Mathematics Newsletter 7 (May-June 1977) 5-10.

Biographical sketch by her granddaughter that includes summary of mathematical contributions together with excerpts from letters.

Pedoe, Dan, Geometry and the Liberal Arts, St. Martin's Pr, 1978; 296 pp, $8.95 (outside of North America: Penguin, 1976, £2.50 (P)).

Brilliant and entertaining elementary survey of geometry's lasting appeal to artists, scientists, and philosophers through the ages. The first few chapters study the attitudes toward geometry of the architect Vitruvius and of the artists Dürer and da Vinci. Later chapters analyze Euclid's *Optics* and *The Elements;* follow up previous discussion of artistic perspective with exposure to projective geometry; and explore form in architecture, unusual plane curves, and the fourth dimension. Diagrams abound, 24 plates are included, and each chapter concludes with a few exercises.

Krouse, John K., *Finite element update,* Machine Design 50 (January 12, 1978) 98-103.

The finite element method is being heavily applied in engineering design. Speedier and cheaper computer algorithms have encouraged this development. This article surveys the state of the art from an engineers point of view.

NEWS & LETTERS

△WELCOME DELTA△

As of January 1978 the undergraduate magazine *Delta* ceased publication and, by agreement with the Mathematical Association of America, was incorporated into *Mathematics Magazine*. We welcome former *Delta* subscribers, and hope they will find in the *Magazine* many of the elements that they found attractive in *Delta*.

In commemoration of this occasion, Steve Bauman, editor of *Delta*, prepared the following capsule history of *Delta*.

In October of 1968, Rajindar S. Luthar, with the encouragement of Dr. Murray Deutsch, then dean of the University of Wisconsin-Waukesha, moved to organize the "Waukesha Mathematical Society." The purpose was to encourage interest in the study of mathematics among the local youth. At a meeting of faculty from U.W.-Waukesha, Carroll College, and local high schools, the society and its journal *Delta* were established. The title of the journal was suggested by Harold Glander of Carroll College. The chief editor was Luthar with Glander and Sister Christopher of Carroll College in associate positions.

The response and encouragement from the mathematical community gradually changed the character of the journal from a local to a national publication. Due to the transfer of some of its members, the Waukesha Mathematics Society faded and subsequently Volumes 2 and 3 of *Delta* were published independently by R.S. Luthar.

In January of 1974 the publication of *Delta* was assumed jointly by the Mathematics Departments of the University of Wisconsin-Madison and University of Wisconsin-Extension. From January 1974 to January 1978 Volumes 4-7 were published in Madison. The editorial board consisted of Steven F. Bauman as chief editor and James Hall as managing editor. Richard A. Brualdi, Donald W. Crowe, and R.S. Luthar were associate editors.

THREE IN ONE

Magazine subscribers will receive this January 1978 issue nearly six months after it was supposed to be published. And then in short order they will also receive the March and May issues. We apologize for these unprecedented delays, and hope that by mid-summer we will be completely back on our regular schedule.

The major difficulty was due to extensive delays at Jerusalem Academic Press where the galleys were composed. These delays began to accumulate late last year to the extent that galley composition took three or four months instead of the normal one month. Because Jerusalem Academic Press was unable to remedy these delays, the Mathematical Association of America shifted their contract for both the *Monthly* and the *Magazine* to Science Typographers in New York. We expect that we will return to a normal schedule as soon as the transition to Science Typographers has been completed.

QUESTIONS WANTED

Two preliminary editions of "Open Questions in Mathematics" have appeared and further contributions are still being sought. The collection consists of "favorite problems" (excluding already well publicized ones) with their histories, hints for solutions, references to past progress, and short biographies of contributors. Manuscripts should not exceed two or three pages and should be submitted in final form.

Requests for copies of the second preliminary edition and submissions should be sent to the editor:

Dagmar R. Henney
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GO FORTH AND MULTIPLY

The result of David Friedman's "Multiplicative Magic Squares" (this *Magazine*, November 1976, pp. 249-250) certainly would be more interesting if the 2 main diagonals also had the same product as the rows and columns. An example of one such p-magic 3x3 square (called a geometric magic square) would be

b	a^2b^2	a
a^2	ab	b^2
ab^2	1	a^2b^2

where a and b are any numbers. An extremely simple way of constructing any size p-magic square is to construct an additive magic square of the desired size and then to use these numbers as exponents to some base.

Jerry Kramer
Northeast High School
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The magic of magic squares extends to more than just multiplication and division. In fact if $*$ is any commutative associative operation with an inverse, e.g., $x*y = x + y + xy$, then the square

a	b	$a^{-1}*b^{-1}*e^{3}$
$a^{-2}*b^{-1}*e^{4}$	e	$a^{2}*b*e^{-2}$
$a*b*e^{-1}$	$b^{-1}*e^{2}$	$a^{-1}*e^{2}$

is magic with respect to $*$ with magic number $e^{3} = e*e*e$. Direct computation shows that for any magic square its magic number is the "cube" of the number at its center with the rest of the elements related as above.

K.R.S. Sastry
Addis Ababa
Ethiopia

In a geometric magic square each prime in the magic product must appear the same number of times in each row, column, and diagonal. It is not difficult to generate a large number of nxn ($n \geq 3$) arrangements with a given prime occurring exactly once in each row, column, and diagonal and with all other entries one (3x3 arrangements require the use of powers of primes). To produce a geometric magic square we need only overlay several of these patterns until there is at least one entry in each cell and then take the product of the entries in each cell. By overlaying several basic patterns with distinct primes, one obtains a geometric magic square with distinct entries. For example if a, b, c, d, e, f, g are distinct primes the seven basic distributions (one for each prime) for a 4x4 square may be overlaid to obtain the following geometric magic square

ag	ce	df	b
bf	d	cg	ae
c	af	be	dg
de	bg	a	cf

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Editor's Note: David Friedman's "Multiplicative Magic Squares" prompted a number of readers to (re)discover the idea of a geometric magic square. The pattern presented by Jerry Kramer was the most popular, and was reported independently in the *Journal of Recreational Mathematics*, Vol. 9, No. 2, p. 129.

COMPUTING CONFERENCE

The Ninth Conference on Computing in the Undergraduate Curriculum (CCUC/9) will be held June 12 at the University of Denver. All disciplines will be represented, including Agriculture, Biology, Business, Chemistry, Earth Sciences, Fine Arts, History, Humanities, Music, Physics, Social Sciences, and Statistics. For further information, contact the conference chairman, Professor William S. Dorn, Department of Mathematics, Univ. of Denver, Denver, CO 80208.

1977 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A-1. Consider all lines which meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points, say

$$(x_i, y_i),\ i = 1, 2, 3, 4 .$$

Show that $(x_1+x_2+x_3+x_4)/4$ is independent of the line and find its value.

A-2. Determine all solutions in real numbers x, y, z, w of the system

$$x + y + z = w,$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w} .$$

A-3. Let u, f, and g be functions, defined for all real numbers x, such that

$$\frac{u(x+1) + u(x-1)}{2} = f(x)$$

and

$$\frac{u(x+4) + u(x-4)}{2} = g(x).$$

Determine $u(x)$ in terms of f and g.

A-4. For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

as a rational function of x.

A-5. Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

for all integers p, a, and b with p a prime, $p > 0$, and $a \geq b \geq 0$.

A-6. Let $f(x,y)$ be a continuous function on the square

$$S = \{(x,y): 0 \leq x \leq 1,\ 0 \leq y \leq 1\} .$$

For each point (a,b) in the interior of S, let $S_{(a,b)}$ be the largest square that is contained in S, is centered at (a,b), and has sides parallel to those of S. If the double integral $\iint f(x,y)\,dx\,dy$ is zero when taken over each square $S_{(a,b)}$, must $f(x,y)$ be identically zero on S?

B-1. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} .$$

B-2. Given a convex quadrilateral $ABCD$ and a point O not in the plane of $ABCD$, locate point A' on line OA, point B' on line OB, point C' on line OC, and point D' on line OD so that $A'B'C'D'$ is a parallelogram.

B-3. An (ordered) triple (x_1, x_2, x_3) of positive *irrational* numbers with $x_1 + x_2 + x_3 = 1$ is called "balanced" if each $x_i < 1/2$. If a triple is not balanced, say if $x_j > 1/2$, one performs the following "balancing act"

$$B(x_1, x_2, x_3) = (x'_1, x'_2, x'_3) ,$$

where $x'_i = 2x_i$ if $i \neq j$ and $x'_j = 2x_j - 1$. If the new triple is not balanced, one performs the balancing act on it. Does continuation of this process always lead to a balanced triple after a finite number of performances of the balancing act?

B-4. Let C be a continuous closed curve in the plane which does not cross itself and let Q be a point inside C. Show that there exists points P_1 and P_2 on C such that Q is the midpoint of the line segment P_1P_2.

B-5. Suppose that $a_1, a_2, \ldots, a_n$ are real ($n>1$) and

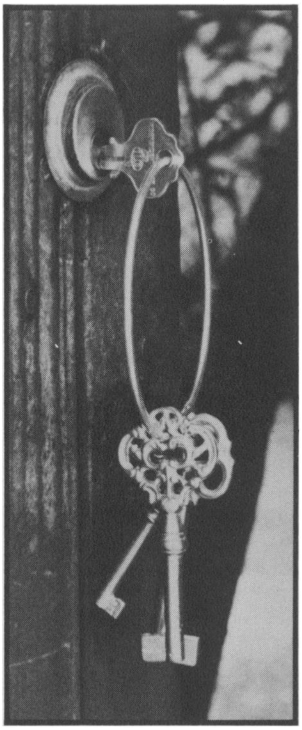
$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2 .$$

Prove that $A < 2a_ia_j$ for $1 \leq i < j \leq n$.

B-6. Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a\ldots x_na$, with n a positive integer and the x_i in H.

(a) Show that P is a finite set.

(b) Show that, in fact, P has no more than $3h^2$ elements.



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